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System failure probability analysis of cohesive slope considering the spatial variability of undrained shear strength

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Abstract: A system failure probability analysis method of cohesive slope considering the spatial variability of undrained shear strength is proposed. In this method, the local averaging parameter of the random field of undrained shear strength on slip surface is introduced as an equivalent parameter. The statistical characters of the equivalent parameter and the correlation coefficient between different equivalent parameters are formulated. Then, the reliability index of a single failure mode and the correlation coefficient between different failure modes are calculated based on the equivalent parameters. By considering both the reliability index and correlation coefficient between different failure modes, the representative slip surfaces are searched step by step, and the system failure probability is assessed using those representative slip surfaces. Finally, to verify this method, three slopes are analyzed as examples. The results show that the equivalent parameter obtained by local averaging along the circular slip surface is feasible to describe the spatial variability of the undrained shear strength, and the proposed method can assess the system failure probability of cohesive slope with small error. Meanwhile, the correlation coefficient between failure modes increases with the spatial correlation of the random field, thus the number of representative slip surfaces required to achieve convergence will reduce.

Keywords: slope stability; system failure probability; spatial variability; correlative failure mode; equivalent parameter

1 Introduction

The reliability analysis of slope system considering the spatial variability of soil parameters has been attracting lots of researchers in slope engineering in recent years. The spatial variability of soil parameters brings certain uncertainty to the failure mode and stability factor of the slope. A single failure mode and deterministic safety factor cannot describe this uncertainty. It is necessary to use the reliability method to calculate the probability of failure of slopes. In the slope reliability analysis, there are countless potential sliding surfaces, and the failure of any one of the potential sliding surfaces can lead to failure of the slope system. In other words, the slope system can be regarded as a series system composed of countless potential sliding surfaces (i.e. failure modes)[1−2].

In the reliability analysis of slope system, the determination of the critical failure slip surface is a difficult problem, especially when the spatial variability of soil parameters is considered. At present, there are mainly two categories of methods to address the critical failure sliding surface. The first category searches for the most dangerous sliding surface every time to ensure that the slope always slides along the sliding surface corresponding to the minimum safety factor, including stochastic finite element method[3−4], stochastic limit equilibrium method[5], and limit analysis method[6−7]. When considering the spatial variability of soil parameters, those methods require a discretization of random field for each simulation. To obtain the failure probability, thousands of Monte Carlo simulations are carried out, and a deterministic analysis is performed to calculate the minimum safety factor of the slope in each simulation. This process commonly requires enormous computing resources. The second category of method selects several representative sliding surfaces as the research object and approximates the slope system into a surrogate system composed of these representative sliding surfaces. In this way, the reliability of the slope system can be estimated by calculating the reliability of the surrogate system[1, 8−9]. This method recognizes that although different sliding surfaces have different positions, they have common (or related) soil parameters, and their safety factors are also related. Zhang et al.[1] showed that if the safety factors of a group of sliding surfaces were highly correlated, their limit state equations were approximately parallel in the standard normal space. According to the geometric meaning of the reliability index, the reliability of the series system composed of this group of sliding surfaces is approximately equal to the minimum reliability of this group of sliding surfaces. On this condition, the sliding surface corresponding to the minimum reliability is a representative sliding surface. The representative sliding surface method improves the efficiency of reliability analysis of slope system, and it has been

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widely used[10–12].

When considering the spatial variability, it is necessary to discretize the random field of soil parameters into many random variables, which leads to a large number of input parameters of the slope system, and it is relatively difficult to calculate the correlation coefficient between failure modes. Currently, the usual approach is to select only the minimum reliability sliding surface to approximate the slope system[13–14]. This method tends to overestimate the reliability of the slope. Another approach is to ignore the spatial variability of the soil in the horizontal direction and only consider its vertical spatial variability. On this condition, the input parameters of the slope system are controllable, and the correlation coefficient between failure modes can also be obtained. Based on this assumption, Zheng et al.[8] used the approximate correlation coefficients and Pearson correlation coefficients to calculate the system reliability of slopes under related failure modes by a discretization of one-dimensional random field. Li et al.[9] considered the vertical spatial variability of soil parameters and discretized the one-dimensional random field to obtain the reliability of the slope by using risk clustering method to identify representative sliding surfaces. However, when the spatial variability of soil parameters in the horizontal direction cannot be ignored, there are many parameters involved in identifying representative sliding surfaces. In this context, it may be very difficult to calculate the correlation coefficients between various failure modes using the above methods.

In this paper, the saturated clay slopes are discussed. The undrained shear strength S_u is locally averaged on the circular sliding surface, the equivalent parameters and their statistical characteristics after the local average are obtained, and the correlation coefficient between the equivalent parameters of different sliding surfaces is derived, thus the correlation coefficient between different failure modes is obtained. By comprehensively considering the correlation coefficient between the reliability index of the potential sliding surface and the failure mode, a method for searching the representative sliding surface step-by-step is developed, and the system failure probability of the slope is calculated. Finally, the method proposed in this paper is used to analyze three slope examples and the results are compared with the previous studies. The results verify the reliability of the proposed method in this paper.

2 Equivalent parameters considering the local average effect

2.1 Variance reduction coefficient of equivalent parameters

For a saturated clay slope shown in Fig.1, the safety factor is

$$F_S = \frac{\sum S_{ui}\Delta l_i}{\sum \Delta W_i \sin\alpha_i} \tag{1}$$

where S_{ui} is the undrained shear strength at the bottom of the soil slice; Δl_i is the length at the bottom side of the soil slice; ΔW_i is the weight of the soil slice; and α_i is the inclination angle of the soil slice.

When considering the spatial variability, a discretization of random field is often needed to obtain the soil parameters at the bottom of the soil slice[14]. This process requires a lot of computing resources. Suchomel et al.[15] and Hu et al.[16] believed that the random field parameters can be locally averaged on the sliding surface, and the equivalent parameters after the local average can be used to transform the spatially variable soil into an equivalent homogeneous soil. This research will continue adopting this idea and introduce this equivalent parameter into the circular sliding surface.

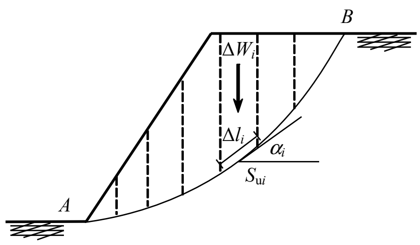

Fig. 1 Bishop method for cohesive slope

In the slope profile shown in Fig.2, $P(x, y)$ is a random field of soil parameters, and its mean and variance are μ and σ^2. The radius of the arc sliding surface L is R, the center coordinates are (X, Y). The polar coordinate system is established with the center of the circle, and the polar angles corresponding to the two ends of the arc are θ_1 and θ_2. The equivalent parameter P_L after the random field is locally averaged on the sliding surface is defined as a curve integral:

$$P_L = \int_L P(x, y)\mathrm{d}s / L \tag{2}$$

where L is the length of the arc curve. According to this definition, combined with Eq.(1), it can be found that the factor of safety for saturated cohesive soil slopes has a linear relationship with the equivalent parameters of undrained shear strength. Calculating the mean and variance of the equivalent parameter P_L, we can get

$$\left.\begin{aligned} E[P_L] &= \mu \\ D[P_L] &= \sigma^2 \cdot \gamma(\theta_1, \theta_2, R, \rho) \end{aligned}\right\} \tag{3}$$

where γ is the variance reduction coefficient of the equivalent parameter; ρ is the correlation function of the random field. Two commonly used correlation functions are exponential correlation function and Gaussian correlation function. For the exponential correlation function, we have

$$\rho(\tau_1,\tau_2)=\exp\left[-2\left(\frac{|\tau_1|}{\delta_1}+\frac{|\tau_2|}{\delta_2}\right)\right] \tag{4}$$

For the Gaussian correlation function, we have

$$\rho(\tau_1,\tau_2)=\exp\left[-\pi\left(\left(\frac{\tau_1}{\delta_1}\right)^2+\left(\frac{\tau_2}{\delta_2}\right)^2\right)\right] \tag{5}$$

After derivation (see appendix for details), the variance reduction coefficient γ is calculated as

$$\left.\begin{aligned}&\gamma(\theta_1,\theta_2,R,\rho)=\int_{\theta_1}^{\theta_2}\int_{\theta_1}^{\theta_2}\rho(\xi_1,\xi_2)\mathrm{d}\alpha_1\mathrm{d}\alpha_2/(\theta_2-\theta_1)^2\\&\xi_1=R(\cos\alpha_1-\cos\alpha_2)\\&\xi_2=R(\sin\alpha_1-\sin\alpha_2)\end{aligned}\right\} \tag{6}$$

It can be seen that the local average effect of the random field does not affect the mean value of the equivalent parameters, but its variance should be reduced on the basis of the "point" variance of the random field. The degree of reduction depends on the correlation function of the random field, the position and the length of the sliding surface. It is worth noting that when an exponential correlation is taken to characterize the random field, the calculation formula of the variance reduction coefficient obtained here is equivalent to that in the literature [16].

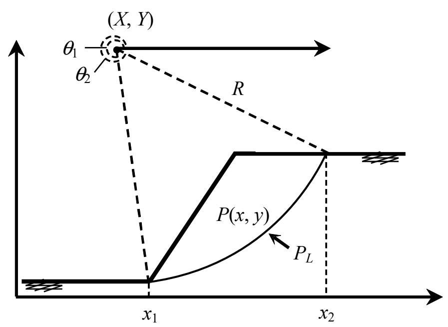

Fig. 2 The equivalent parameter of circular slip surface

2.2 Correlation coefficients between equivalent parameters

Assuming two sliding surfaces L_1 and L_2 in Fig.3, the center coordinates of the sliding surface L_1 are (X_1, Y_1), the radius is R_1, the polar angles of the two arc end points in polar coordinates are θ_{11} and θ_{12}, respectively. The same applies to the sliding surface L_2. The equivalent parameters of the random field $P(x, y)$ after being locally averaged on two arcs are P_{L1} and P_{L2}, respectively, and the correlation coefficient between the two equivalent parameters is

$$\left.\begin{aligned}&r_{12}=\int_{\theta_{11}}^{\theta_{12}}\int_{\theta_{21}}^{\theta_{22}}\rho(\xi_1,\xi_2)\mathrm{d}\alpha_1\mathrm{d}\alpha_2/\left[\sqrt{\gamma_1\gamma_2}(\theta_{22}-\theta_{21})(\theta_{12}-\theta_{11})\right]\\&\xi_1=X_1+R_1\cos\alpha_1-X_2-R_2\cos\alpha_2\\&\xi_2=Y_1+R_1\sin\alpha_1-Y_2-R_2\sin\alpha_2\end{aligned}\right\} \tag{7}$$

where γ_1 and γ_2 are the variance reduction coefficients of the equivalent parameters of the sliding surface respectively, and the derivation process is elaborated in the appendix.

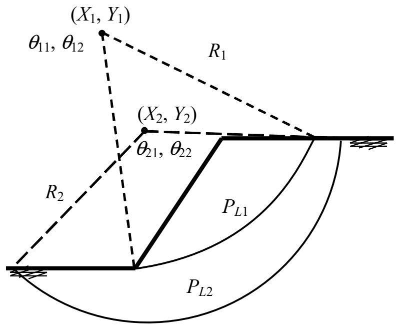

Fig. 3 Correlation coefficient between two equivalent parameters

3 Reliability of undrained slope system

3.1 Response surface method based on equivalent parameters

As mentioned above, by introducing the equivalent parameters of undrained shear strength, the safety factor of saturated clay slope for a given sliding surface is a linear function of the equivalent parameters. A linear function can therefore be used as the performance function of the sliding surface, namely

$$g(\boldsymbol{X})=F_{\mathrm{S}}-1=\sum_{i=1}^{n}a_ix_i-1 \tag{8}$$

where g is the response surface function; $\boldsymbol{X}=[x_1, x_2, \cdots, x_n]^{\mathrm{T}}$ is a vector including the basic variables of the response surface equation; n is the number of variables, which generally equals to the number of soil layers where the sliding surfaces passing through; a_i is parameters to be determined. There are a total of n undetermined parameters in Eq.(8). In order to determine these n undetermined parameters, n^{th} sampling calculations are required, and the points sampled at the mean values of the equivalent parameters are sufficient to meet the requirements.

We take a two-layered slope as shown in Fig.4 as an example to briefly explain the application of equivalent parameters in the response surface equation. In the figure, the sliding surfaces S_i and S_j pass through two layers of soil, and their corresponding equivalent parameters in the two layers of soil are $c_i^{(1)}$, $c_i^{(2)}$and $c_j^{(1)}$, $c_j^{(2)}$, respectively.

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The superscripts represent the soil layer, and the subscripts represent the sliding surface. Therefore, the response surface functions of the sliding surface are

$$\left.\begin{aligned} g_i &= g_i(c_i^{(1)}, c_i^{(2)}) \\ g_j &= g_j(c_j^{(1)}, c_j^{(2)}) \end{aligned}\right\} \tag{9}$$

When the parameters of different soil layers are independent of each other, their corresponding equivalent parameters, such as $c_i^{(1)}$ and $c_i^{(2)}$, are also independent of each other. However, the equivalent parameters in the same soil layer, such as $c_i^{(1)}$ and $c_j^{(1)}$, are related, and the correlation coefficient can be calculated by Eq.(7).

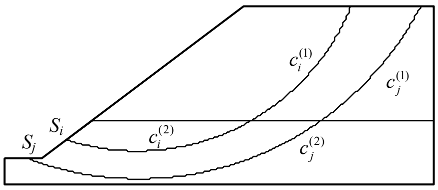

Fig. 4 Two illustrative slip surfaces and its equivalent parameters of a two-layered soil slope

The first-order reliability method (FORM) is used to calculate the reliability index of the slope in single failure mode. According to the classic FORM theory, the reliability index β is the shortest distance from the origin to the limit state surface in the standard normalized space[17], namely

$$\beta = \min_{g=0} \sqrt{\boldsymbol{\alpha}^{\mathrm{T}} \boldsymbol{\alpha}} \tag{10}$$

where $g = 0$ is the limit state equation; $\boldsymbol{\alpha}$ is the independent standard normalized variables in the limit state equation. When the minimum value is obtained on the right side of the equation, the minimum point (denoted as $\boldsymbol{\alpha}^*$) is the design point. In this study, all variables are lognormal distributions, which can be converted to normal distributions ln($\boldsymbol{X}$) by taking the natural logarithm, and then ln($\boldsymbol{X}$) is converted to standard normal distribution space by taking $(\ln(\boldsymbol{X}_i)-\boldsymbol{\mu}_i)/\boldsymbol{\sigma}_i$. Since the transformation from lognormal distribution to normal distribution is nonlinear, the correlation coefficient between random variables has changed, but the difference is small, and the influence on the calculation result is negligible[18]. As a result, the influence is not taken into account in this research. At the same time, the optimization problem described by Eq.(10) can be solved by the built-in function 'fmincon' in MATLAB.

3.2 Correlation coefficients between different failure modes

Suppose the design points of two failure modes $g_i \leqslant 0$ and $g_j \leqslant 0$ (denoted by F_i and F_j, similarly hereinafter) are $\boldsymbol{\alpha}_i^*$ and $\boldsymbol{\alpha}_j^*$, respectively. When the spatial variability of soil parameters is not considered, the corresponding parameters in $\boldsymbol{\alpha}_i^*$ and $\boldsymbol{\alpha}_j^*$ are completely related, which is the situation in the literatures[1, 19]. At this moment, the correlation coefficient between the two failure modes can be calculated using the following formula[19]:

$$\lambda_{ij} = \frac{\boldsymbol{\alpha}_i^{*\mathrm{T}} \boldsymbol{\alpha}_j^*}{\beta_i \beta_j} \tag{11}$$

However, when considering the spatial variability, the parameters corresponding to the design points $\boldsymbol{\alpha}_i^*$ and $\boldsymbol{\alpha}_j^*$ are not completely correlated. For example, for $c_i^{(1)}$ and $c_j^{(1)}$ in Eq.(9), their correlation coefficient is described by Eq.(7). Because of this correlation, the correlation coefficient between the two failure modes should be smaller than when spatial variability is not considered. The correlation coefficient between the two failure modes after introducing equivalent parameters is deduced as follows:

$$\lambda_{ij} = \frac{\boldsymbol{\alpha}_i^{*\mathrm{T}} \left[r_{ij}^{(kk)} \right] \boldsymbol{\alpha}_j^*}{\beta_i \beta_j} \tag{12}$$

where $[r_{ij}^{(kk)}]$ is a diagonal matrix, and the elements on the diagonal $r_{ij}^{(kk)}$ represent the correlation coefficient between the equivalent parameters $x_i^{(k)}$ and $x_j^{(k)}$ after the local average of the random field under the failure modes F_i and F_j. The superscript k represents the serial number of soil layer, and the subscripts i and j represent the serial number of sliding surface. Similarly, when the parameters of different soil layers are independent of each other, their corresponding equivalent parameters are also related and independent, whereas the equivalent parameters in the same soil layer are interrelated, so $[r_{ij}^{(kk)}]$ is a diagonal matrix.

$$\left[r_{ij}^{(kk)} \right] = \begin{bmatrix} r_{ij}^{(11)} & & & \\ & r_{ij}^{(22)} & & \\ & & \cdots & \\ & & & r_{ij}^{(nn)} \end{bmatrix} \tag{13}$$

3.3 Calculation of system failure probability

As mentioned above, the slope system is a series system composed of countless potential sliding surfaces, and the system failure probability can be approximated by several (set to N) related representative failure modes. Suppose the reliability index of these N representative failure modes F_k (k = 1, 2, …, N) is $\boldsymbol{\beta} = [\beta_1, \beta_2, \ldots, \beta_N]$, the correlation coefficient matrix between the failure modes is $\boldsymbol{\Lambda}$, then the failure probability calculation of the surrogate system is actually to find the value of the cumulative probability function of the multivariate normal distribution, namely

$$_f(\bigcup_{k=1}^{N} F_k) = 1 - P_f(\overline{\bigcup_{k=1}^{N} F_k}) = 1 - P_f(\bigcap_{k=1}^{N} \overline{F_k}) = 1 - \Phi(\boldsymbol{\beta}, \boldsymbol{\Lambda}) \tag{14}$$

where $\cup$ represents the union of events; $\cap$ represents

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the intersection of events; $\overline{F_k}$ represents the complement set of event F_k; and Φ is the cumulative probability function of the multivariate normal distribution, its mean value is 0, and the covariance matrix is $\boldsymbol{\Lambda}$.

The key issue is how to select representative sliding surfaces from a large number of potential sliding surfaces. Zhang et al.[1] took the minimum reliability sliding surface as the first representative sliding surface, then pre-selected the correlation coefficient threshold between failure modes λ_0, and searched for the representative sliding surface by gradually eliminating the sliding surfaces which were highly correlated with the current representative sliding surface ($\lambda > \lambda_0$). When the spatial variability is not considered, soil parameters are completely correlated in the same soil layer, thus the values of the correlation coefficients between failure modes are commonly large. According to the analysis of Zhang et al.[1], for a simple homogeneous slope, the convergence results can be obtained when the correlation coefficient threshold λ_0 is set to 0.6. However, when considering the spatial variability, the correlation coefficient between different failure modes is smaller since the correlation coefficient of equivalent parameters between different sliding surfaces is less than 1 (Eq.(12)). Consequently, in order to make the calculated system failure probability converge, the correlation coefficient threshold λ_0 should take a relatively large value, so the number of representative sliding surfaces is bound to increase dramatically. On the other hand, as shown in Eq.(14), the failure probability of the surrogate system composed of representative sliding surfaces is related to the reliability index and the correlation coefficient matrix. Hence, it is a more reasonable method to comprehensively consider the reliability indexes of potential failure modes and their correlation coefficients to search for representative sliding surfaces.

Based on the above considerations, this paper proposes a representative sliding surface searching method based on optimization analysis. This method first takes the sliding surface with the minimum reliability as the first representative sliding surface, and the next representative sliding surface should maximize the system failure probability of the new surrogate model, namely

$$\max_{S_1} P_{f_1} = 1 - \Phi_0(\beta_1) \tag{15}$$

$$\max_{S_k} P_{f_k} = 1 - \Phi\left\{\left[\boldsymbol{\beta}_{k-1}, \beta_k\right], \begin{bmatrix} \boldsymbol{\Lambda}_{k-1} & \boldsymbol{\lambda}_{k-1} \\ \boldsymbol{\lambda}_{k-1}^{\mathrm{T}} & 1 \end{bmatrix}\right\} \tag{16}$$

where S_1 is the first representative sliding surface; β_1 is the reliability index of the first representative sliding surface; Φ_0 is the cumulative distribution function of the standard normal distribution; k = 2, 3, 4, …, which means the number of representative sliding surfaces; S_k represents the k-th representative sliding surface; $\boldsymbol{\beta}_{k-1}$ and $\boldsymbol{\Lambda}_{k-1}$ are the reliability index vector and correlation coefficient matrix of the first $k-1$ representative sliding surfaces; β_k is the reliability index of the k-th representative sliding surface; and $\boldsymbol{\lambda}_{k-1}$ is the correlation coefficient vector describing the correlations between the k-th representative sliding surface and the first $k-1$ representative sliding surface.

In the process of searching for representative sliding surfaces, the failure probability of the surrogate system gradually converges as the number of representative sliding surfaces increases. If the difference between the previous failure probability and the next failure probability is less than a given allowable error ε, the entire searching process can be ended. The calculation process of searching the representative sliding surface step by step is as follows:

(1) Search for the sliding surface of the minimum reliability according to Eq.(15) and take it as the first representative sliding surface to calculate the failure probability of the current surrogate system.

(2) Search for the next representative sliding surface according to Eq.(16) to obtain the failure probability of the new surrogate system, and compare it with the failure probability of the previous one to calculate the relative error.

(3) If the relative error is greater than the given allowable error, repeat step (2), otherwise the calculation ends.

According to the description of Eqs. (15) and (16), the searching problem of representative sliding surfaces is actually an optimization problem, that is, to determine a sliding surface so that Eq.(15) (or Eq.(16)) reaches the maximum value. Under the assumption of circular sliding surface, the sliding surface is completely characterized by its center coordinates X, Y and radius R. At this time, the search for representative sliding surfaces is to determine three variables (X, Y, R), which lead to that the objective function described in Eqs. (15) and (16) takes the maximum value. This research uses genetic algorithm to solve this problem. The following takes the minimum reliability sliding surface searching problem as an example to briefly describe the calculation process of the genetic algorithm:

(1) Randomly generate several potential sliding surfaces (that is, several sets of control variables X, Y, R), calculate the equivalent parameters of the corresponding sliding surfaces, establish the response surface equation, and use Eqs. (11) and (15) to calculate its reliability index and value of the objective function.

(2) Select a certain proportion of sliding surfaces with high adaptability (large objective function value) from the parent sliding surfaces, and perform crossover, mutation

and recombination operations on its control parameters (X, Y, R) to obtain a new set of sliding surfaces and calculate its objective function value.

(3) Repeat step (2) until the number of iterations reaches the predetermined maximum number of generations.

In the iterative process, due to the operations such as crossover, mutation, and recombination, the excellent genes of the previous generation have a greater probability of being passed on to the next generation, so that the optimal individual in the next generation gradually approaches the optimal solution of the objective function, that is, the search gives the sliding surface with the smallest reliability. In the same way, when searching the following representative sliding surfaces, the same procedure is performed, while the only difference is that the objective function is replaced by Eq.(16). More information on genetic algorithms can be found in literature [20].

4 Example analysis

4.1 Case 1

Consider a single-layered slope as shown in Fig.5. The calculation model and parameters are selected from literature [5]. The slope height is 5 m, the slope is 1 : 2, and the soil unit weight is 20 kN /m^3. The undrained shear strength of soil S_u is a lognormal distribution, and its mean value and coefficient of variation are 23 kPa and 0.3. The exponential correlation function is used to describe the spatial correlation of S_u, and the horizontal and vertical correlation lengths are δ_1 = 40 m and δ_2 = 4 m. The Bishop method is used to calculate the safety factor of the sliding surface of the slope. The deterministic analysis result shows that the safety factor of the slope is 1.357, which is consistent with the result (1.356) from the literature [5]. Since S_u is lognormally distributed, its equivalent parameter is the sum of countless lognormal distributions. Its true distribution is between the normal distribution and the lognormal distribution and is closer to the lognormal distribution[21]. This article assumes that the equivalent parameters obey the lognormal distribution and analysis is performed based on this.

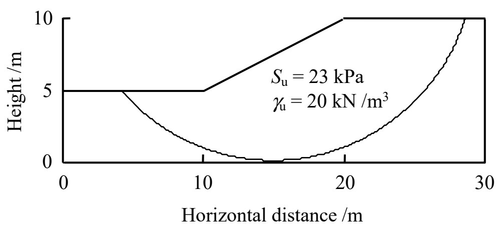

Fig. 5 Case 1 a simple slope (F_S = 1.357)

The failure probability of the deterministic sliding surface is denoted as $P_{f\text{-F}}$. The calculation results show that the failure probability of the deterministic sliding surface is 0.033 0, which is in good agreement with the 0.0316 given in the literature [5].

The system failure probability (denoted as $P_{f\text{-S}}$) is calculated by gradually searching the representative sliding surface of the slope, and the convergence condition is set to ε = 1%. After 13 searches, the loop is terminated, and 13 representative sliding surfaces are obtained. At this time, the slope failure probability is 0.072 4, which is slightly smaller than the result (0.076 0) reported in the literature [5]. Figure 6 shows the convergence curve of the failure probability of the slope system. When there is only one representative sliding surface (that is, the critical probability sliding surface), the slope failure probability is 0.033 4. As the number of representative sliding surfaces increases, the failure probability of the surrogate system also increases but gradually converges to the overall failure probability of the slope system.

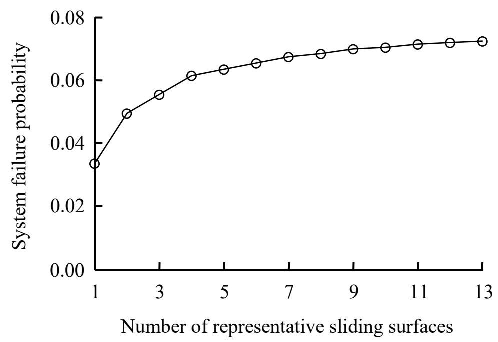

Fig. 6 Convergence of system failure probability(case 1)

Figure 7 shows the locations of the first four representative sliding surfaces. The failure probabilities of these four representative sliding surfaces are 0.033 4, 0.027 3, 0.017 3, and 0.032 2. The correlation coefficient matrix among them is

$$\boldsymbol{\Lambda}=\begin{bmatrix}1 & 0.734 & 0.562 & 0.901\\0.734 & 1 & 0.847 & 0.877\\0.562 & 0.847 & 1 & 0.683\\0.901 & 0.877 & 0.683 & 1\end{bmatrix} \qquad (17)$$

It can be noted that because the geometric positions of S_1 and S_4 are relatively close, the correlation of the soil parameters is relatively strong, resulting in the numerical value of the failure probability being relatively close, and the correlation between the two failure modes is also relatively strong (λ_{14} = 0.901). On the other hand, although the failure probability of S_4 is greater than those of S_2 and S_3, its correlation with S_1 is too strong, resulting in its contribution to the failure probability of the surrogate system is not as good as S_2 and S_3. Meanwhile, although the correlation coefficient between S_3 and S_1 is smaller than the correlation coefficient between S_2 and S_1, the failure probability of S_3 is smaller than that of S_2, which leads to a greater contribution of S_2 to the failure probability of the surrogate system than that of S_3. This phenomenon demonstrates that when selecting representative sliding surfaces, it is more reasonable to comprehensively consider the failure probability (reliability index) and correlation coefficients of each potential sliding surface.

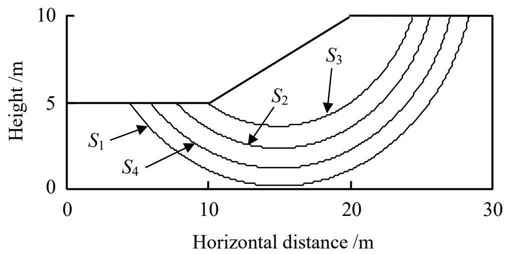

Fig. 7 Locations of the first four representative slip surfaces(case 1)

Table 1 lists the response surface equations of the first four representative sliding surfaces and their analysis results, where c is the equivalent parameter of the undrained shear strength of the soil. Perform three types of Monte Carlo simulations on these 4 representative sliding surfaces (1×10^5 times): i) Equivalent parameters + response surface method; ii) Equivalent parameters + Bishop method; iii) Random field discretization + Bishop method. The first two types of Monte Carlo simulation are used to verify the applicability of response surface equations, and the third type of Monte Carlo simulation is used to verify the applicability of equivalent parameters. When discretizing the random field, the shear strength parameters at the midpoint of the bottom of the soil slices on the sliding surface are sampled. Since there are only 4 representative sliding surfaces, each sliding surface can be divided into a number of slices to obtain the coordinate set of the bottom midpoints of the four sliding surface slices, and then sampling by covariance matrix decomposition method to obtain the random field parameters at corresponding locations of soil slices. The thickness of the soil slice determines the discretization accuracy of the random field, and the thickness of the soil slice here is taken as 0.1 m.

Table 1 The results of the first four representative slip surfaces(case 1)

Representative slip surface	Response surface equation $g_i(X)$	Variance reduction factor γ_X	$\mu_{\ln(X)}$	$\sigma_{\ln(X)}$	Probability of failure		
					MCS：Equivalent parameters + Response surface method	MCS：Equivalent parameters + Bishop	MCS：Random field discretization + Bishop
S_1	$g_1 = 0.059\,0c - 1$	0.286 4	3.123	0.159 5	0.033 2	0.033 2	0.031 7
S_2	$g_2 = 0.062\,3c - 1$	0.362 1	3.120	0.179 1	0.027 4	0.027 4	0.025 7
S_3	$g_3 = 0.066\,4c - 1$	0.415 2	3.117	0.191 5	0.017 4	0.017 4	0.016 1
S_4	$g_4 = 0.060\,2c - 1$	0.318 9	3.121	0.168 2	0.032 3	0.032 3	0.030 5

It can be seen from the table that the calculation results of failure probability using equivalent parameters (Monte Carlo simulation based on the vertical slice method and Monte Carlo simulation based on the response surface method) are very close, which demonstrates that the response surface equation is applicative to be a surrogate model of the vertical slice method. On the other hand, the failure probability obtained by the discretization of random field using Monte Carlo simulation is slightly smaller than the first three. This result shows that the equivalent parameters may still lead to certain errors, but it is generally acceptable. By analyzing the four sets of safety factors obtained by the Monte Carlo simulation of the random field and calculating the Pearson correlation coefficient matrix, we can obtain

$$\boldsymbol{\Lambda}' = \begin{bmatrix} 1 & 0.733 & 0.560 & 0.901 \\ 0.733 & 1 & 0.847 & 0.876 \\ 0.560 & 0.847 & 1 & 0.682 \\ 0.901 & 0.876 & 0.682 & 1 \end{bmatrix} \qquad (18)$$

It can be found that this result is almost the same as the correlation coefficient matrix between the first four failure modes calculated in this paper, which also verifies the correctness of the method in this paper.

By changing the horizontal and vertical fluctuation range of the random field, the analysis is performed again using the method of this paper, and the results are summarized in Table 2. It can be seen that as the correlation of the random field increases (the correlation length increases), the number of representative sliding surfaces satisfying the same accuracy ($\varepsilon = 1\%$) decreases, and the slope failure probability increases. From Eq.(12), it is concluded that a stronger correlation enhances the correlation between failure modes, which leads to a reduction in the number of representative sliding surfaces required by the surrogate system. Meanwhile, an increase in the correlation length of the random field can lead to an increase in the variance reduction function of the equivalent parameter, that is, a larger value of variance of the equivalent parameter, thus the failure probability increases.

4.2 Case 2

The slope case in Fig.8 is selected from literatures

Table 2 Comparison of results under different correlation lengths(case 1)

δ_1 /m	δ_2 /m	Number of representative sliding surfaces	Probability of failure		Error /%	Type of failure probability
			Results from this article	Results from literature [5]		
40	4	13	0.033 0	0.031 6	4.4	$P_{f\text{-F}}$
			0.072 4	0.076 0	−4.7	$P_{f\text{-S}}$
40	8	10	0.063 0	0.062 1	1.4	$P_{f\text{-F}}$
			0.101 1	0.109 1	−7.3	$P_{f\text{-S}}$
80	4	9	0.044 4	0.042 2	5.2	$P_{f\text{-F}}$
			0.084 2	0.089 4	−5.8	$P_{f\text{-S}}$

[9, 22]. The slope height is 10 m, the slope is 1:2, and the soil unit weight is 20 kN /m^3. The undrained shear strength of soil S_u obeys a lognormal distribution, and its mean value and coefficient of variation are 40 kPa and 0.3. The exponential correlation function is employed to characterize the spatial correlation of the random field of soil parameters. Meanwhile, the inhomogeneity of the soil parameters in the horizontal direction is ignored, and only the vertical inhomogeneity is considered, that is, $\delta_1 = \infty$ and $\delta_2 = \delta < \infty$. The deterministic analysis results show that the safety factor of the slope is 1.178, which accords with 1.18 in literature [9] and 1.178 in literature [22].

A series of simulations with different values of vertical correlation length δ are performed by the proposed method, and the failure probabilities are calculated while the results are displayed in Fig.9. It can be observed that the calculation results of this paper are in good agreement with the results of literature [22]. As δ increases, the probability of slope failure increases, but its growth rate gradually decreases.

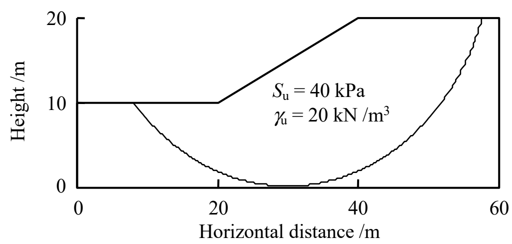

Fig. 8 Case 2 a simple slope (F_S = 1.178)

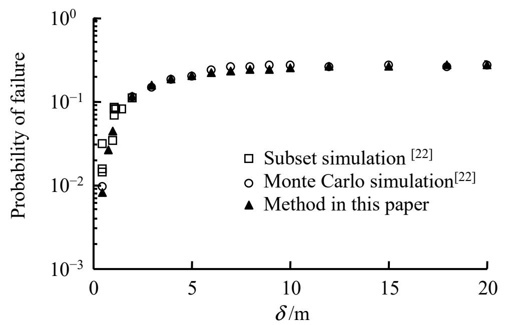

Fig. 9 Effect of vertical correlation length on failure probability(case 2)

When δ = 5 m, nine representative sliding surfaces are identified using the method proposed in this paper. The failure probability of the surrogate system composed of these nine representative sliding surfaces is 0.198 6, which is similar to the 0.197 8 in literature [22] and 0.189 0 in literature [9], and the contribution of the first representative sliding surface to the system failure probability is 60.6%. When δ = 10 m, eight representative sliding surfaces are identified using the method proposed in this paper. The failure probability of the surrogate system composed of these eight representative sliding surfaces is 0.243 2, which is similar to the 0.260 2 in literature [22] and 0.239 0 in literature [9], and the contribution of the first representative sliding surface to the system failure probability reaches 74.1%. It can be found that as δ increases, the correlation of the random field parameters increases, and the contribution of the first representative sliding surface to the failure probability of the slope system increases, and the number of representative sliding surfaces required by the surrogate system also decreases. In fact, for a single-layered undrained slope, when $\delta = \infty$, the correlation coefficient between any two failure modes is 1 according to Eq.(12), and only one representative sliding surface is needed to completely represent the overall failure probability of the slope system.

4.3 Case 3

Figure 10 shows a double-layered undrained slope. In literatures [19, 23−24], random variables were used to describe the uncertainty of soil parameters and the reliability of the slope was studied. Jiang et al.[25] considered the spatial variability of soil parameters and studied the reliability of this slope in a low level of failure probability. As shown in Fig.10, the saturated unit weight of the two layers of soil on the slope is 19 kN /m^3, the undrained shear strength of the first layer of soil is 120 kPa, and the one of the second layer of soil is 160 kPa. The lognormal distribution is used to describe the undrained shear strength of the two layers of soil. The mean value and coefficient of variation of the first layer of soil are 120 kPa and 0.3, and the ones of the second layer of soil are 160 kPa and 0.3. The Gaussian correlation function is used to describe the spatial correlation of the soil parameters, and the horizontal and vertical correlation lengths are taken as δ_1 =

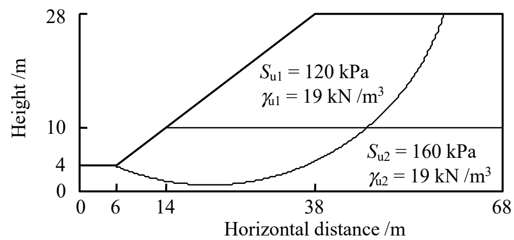

Fig. 10 Case 3 a two-layered slope (F_S = 1.993)

https://rocksoilmech.researchcommons.org/journal/vol42/iss6/2
DOI: 10.16285/j.rsm.2020.6476

40 m and $\delta_2 = 4$ m. The results of deterministic analysis show that the safety factor of the slope is 1.990, which is close to 1.997 in literature [19], 1.992 in literature [24] and 1.993 in literature [25].

The method in this paper is used to calculate the system failure probability of the slope, and the calculation is terminated after five representative sliding surface searches, and the system failure probability of the slope is 1.65×10^{-7}. This result is in the same order of magnitude as 1.14×10^{-7} in the literature [25], but the relative error is large.

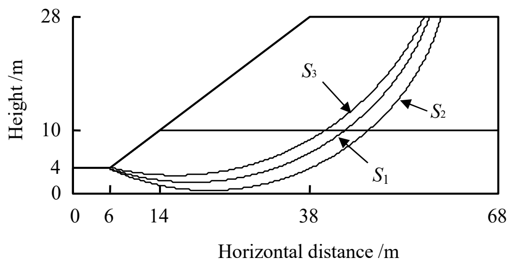

Fig. 11 Locations of the first three representative slip surfaces(case 3, COV = 0.4)

Table 3 The results of the first three representative slip surfaces(case 3, COV = 0.4)

Representative slip surface	Response surface equation $g_i(\boldsymbol{X})$	Variance reduction factor γ_X	$\mu_{\ln(X)}$	$\sigma_{\ln(X)}$	Probability of failure: MCS: Equivalent parameters + Response surface equation	MCS: Equivalent parameters + Bishop	MCS: Random field discretization + Bishop
S_1	$g_1 = 0.004\ 98c_1 + 0.008\ 74c_2 - 1$	$[0.201\ 4, 0.414\ 7]^T$	$[4.771\ 6, 5.043\ 1]^T$	$[0.178\ 1, 0.253\ 5]^T$	4.88×10^{-5}	4.88×10^{-5}	3.10×10^{-5}
S_2	$g_2 = 0.004\ 45c_1 + 0.009\ 11c_2 - 1$	$[0.201\ 7, 0.362\ 0]^T$	$[4.771\ 6, 5.047\ 0]^T$	$[0.178\ 2, 0.237\ 3]^T$	3.47×10^{-5}	3.47×10^{-5}	1.84×10^{-5}
S_3	$g_3 = 0.005\ 70c_1 + 0.008\ 45c_2 - 1$	$[0.202\ 1, 0.457\ 3]^T$	$[4.771\ 6, 5.039\ 9]^T$	$[0.178\ 4, 0.265\ 7]^T$	3.38×10^{-5}	3.38×10^{-5}	1.95×10^{-5}

Re-calculation is conducted using the method in this article by increasing the coefficient of variation of the undrained shear strength of the soil from 0.3 to 0.4, while other parameters are kept unchanged. After searching for five representative sliding surfaces, the calculation is terminated, and the failure probability of the slope is 1.39×10^{-4}. Figure 11 shows the position distribution of the first three representative sliding surfaces. The equivalent parameters, response surface equations and analysis results of these three sliding surfaces are summarized in Table 3, where c_1 and c_2 are equivalent parameters of the undrained shear strength of the first and second layers of soil. Consequently, the correlation coefficients of the three representative sliding surfaces are $\lambda_{12} = 0.891$, $\lambda_{13} = 0.912$ and $\lambda_{23} = 0.696$.

Since the failure probability is in the level of 10^{-5}–10^{-4}, in order to make the Monte Carlo simulation results reliable, 2×10^7 Monte Carlo simulations were performed on these three representative sliding surfaces. When discretizing the random field, the thickness of the soil slice is set to 0.2 m. It can be concluded from the table that, due to the linear relationship between the safety factor and the equivalent parameters, the Monte Carlo simulation based on the vertical slice method and the Monte Carlo simulation based on the response surface method have the same results. By analyzing the three groups (2×10^7 in each group) of safety factors obtained from Monte Carlo simulation integrated with random field discretization and calculating the Pearson correlation coefficient, we can get: $\lambda'_{12} = 0.872$, $\lambda'_{13} = 0.892$ and $\lambda'_{23} = 0.644$. This result is similar to the result in this paper. Furthermore, the failure probability of the sliding surface obtained by the Monte Carlo simulation integrated with random field discretization is smaller than that with the equivalent parameter, which is similar to the results of Table 1.

In literature [19], the spatial variability of soil parameters was not considered. Two representative sliding surfaces (passing only through the first layer of soil and through two layers of soil at the same time) were artificially selected for analysis, and the failure probability of the slope was obtained as 4.02×10^{-3}–4.11×10^{-3}. Using the method in this paper, take $\delta_1 = \infty$ and $\delta_2 = \infty$, after searching for two representative sliding surfaces, calculation terminates due to convergence, and the obtained system failure probability of the slope is 4.20×10^{-3}.

Table 4 Summary of failure probability(case 3)

δ_1 /m	δ_2 /m	COV S_{u1}	COV S_{u2}	Number of representative sliding surfaces	Probability of failure: In this article	From literature[25]	From literature[19]
40	4	0.3	0.3	5	1.65×10^{-7}	1.14×10^{-7}	—
40	4	0.4	0.4	5	1.39×10^{-4}	1.12×10^{-4}	—
∞	∞	0.3	0.3	2	4.20×10^{-3}	4.04×10^{-3}	4.02×10^{-3}–4.11×10^{-3}

5 Conclusion

Based on the equivalent parameters after local averaging of random fields and the first-order reliability method, this paper analyzes the correlation between failure modes of different sliding surfaces, and proposes a reliability method for clay slope considering the spatial variability of undrained strength. The feasibility of the proposed method is discussed by analyzing the failure probability of three slope cases, and the main conclusions are drawn as follows:

(1) The equivalent parameters of the undrained shear strength random field averaged locally along the sliding surface can describe the uncertainty of the soil parameters of the corresponding sliding surface. The correlation between the equivalent parameters has a significant effect on the correlation coefficient between the failure modes. The method proposed in this paper can gradually approximate the system failure probability of the slope.

(2) As the spatial correlation of the random field increases, the correlation coefficient between different failure modes increases, and the contribution of the first representative sliding surface to the system failure probability increases, hence, the number of representative sliding surfaces required by the condition of convergence also decreases. For a single-layered undrained slope, when the spatial variability is ignored, only one representative sliding surface is needed to obtain the system failure probability of the slope.

(3) The method in this paper is only applicable to the case of saturated undrained ($\varphi = 0$) clay slopes. On this condition, the safety factor of the slope has a linear relationship with the equivalent parameters. When $\varphi \neq 0$, the safety factor of the slope and the equivalent shear strength parameter no longer show a linear relationship. It is not clear whether the equivalent parameter after the local average of the random field along the sliding surface can represent the parameter variability of the entire sliding surface. Further analysis is needed to study this problem.

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Appendix

(1) Mean and variance of equivalent parameters

According to the definition of Eq.(2), the equivalent parameter is the local average of the random field parameters on the sliding surface. Therefore,

$$E[P_L]=E\left[\int_L P(x,y)\mathrm{d}s/L\right]=\int_L E\left[P(x,y)\right]\mathrm{d}s/L=\int_L \mu\mathrm{d}s/L=\mu \tag{A1}$$

$$\begin{aligned}D[P_L]&=E\left[\int_L P(x,y)\mathrm{d}s/L-\mu\right]^2=E\left[\int_L P(x,y)\mathrm{d}s/L-\int_L\mu\mathrm{d}s/L\right]^2=E\left[\int_L(P(x,y)-\mu)\mathrm{d}s/L\right]^2=\\&E\left[\int_L(P(x,y)-\mu)\mathrm{d}s\int_L(P(x',y')-\mu)\mathrm{d}s'\right]/L^2=\frac{1}{L^2}\iint_{LL}E[(P(x,y)-\mu)(P(x',y')-\mu)]\mathrm{d}s\mathrm{d}s'\end{aligned} \tag{A2}$$

According to the relationship between the correlation function and the variance function, there are

$$E[(P(x,y)-\mu)(P(x',y')-\mu)]=C\left[(x,y),(x',y')\right]=\sigma^2\rho\left[(x,y),(x',y')\right]=\sigma^2\rho(x-x',y-y') \tag{A3}$$

At the same time, the integration path is expressed in polar coordinates: $x=X+R\cos(\alpha)$, $y=Y+R\sin(\alpha)$, $\mathrm{d}s=R\mathrm{d}\alpha$, substituting these into Eq.(A2), then we can get

$$D[P_L]=\frac{\sigma^2}{(\theta_2-\theta_1)^2}\int_{\theta_1}^{\theta_2}\int_{\theta_1}^{\theta_2}\rho\left[R(\cos\alpha_1-\cos\alpha_2),R(\sin\alpha_1-\sin\alpha_2)\right]\mathrm{d}\alpha_1\mathrm{d}\alpha_2 \tag{A4}$$

(2) Correlation coefficient between equivalent parameters

First calculate the covariance of equivalent parameters, which is similar to Eq.(A2), with

$$\begin{aligned}\mathrm{COV}(P_{L_1},P_{L_2})&=E\left[(P_{L_1}-E(P_{L_1}))(P_{L_2}-E(P_{L_2}))\right]=\frac{1}{L_1L_2}\iint_{L_1L_2}E[(P(x,y)-\mu)(P(x',y')-\mu)]\mathrm{d}s\mathrm{d}s'=\\&\frac{\sigma^2}{(\theta_{12}-\theta_{11})(\theta_{22}-\theta_{21})}\int_{\theta_{11}}^{\theta_{12}}\int_{\theta_{21}}^{\theta_{22}}\rho(X_1+R_1\cos\alpha_1-X_2-R_2\cos\alpha_2,Y_1+R_1\sin\alpha_1-Y_2-R_2\sin\alpha_2)\mathrm{d}\alpha_1\mathrm{d}\alpha_2\end{aligned} \tag{A5}$$

Therefore, the correlation coefficient between the equivalent parameters P_{L_1} and P_{L_2} is

$$\left.\begin{aligned}&r_{12}=\frac{\mathrm{COV}(P_{L_1},P_{L_2})}{\sqrt{D(P_{L_1})}\sqrt{D(P_{L_2})}}=\int_{\theta_{11}}^{\theta_{12}}\int_{\theta_{21}}^{\theta_{22}}\rho(\xi_1,\xi_2)\mathrm{d}\alpha_1\mathrm{d}\alpha_2\Big/\\&\sqrt{\gamma_1\gamma_2}(\theta_{22}-\theta_{21})(\theta_{12}-\theta_{11})\\&\xi_1=X_1+R_1\cos\alpha_1-X_2-R_2\cos\alpha_2\\&\xi_2=Y_1+R_1\sin\alpha_1-Y_2-R_2\sin\alpha_2\end{aligned}\right\} \tag{A6}$$

(3) Correlation coefficient between failure modes

Random variable $\boldsymbol{X}$ is transformed to obtain mutually independent standard normally distributed variable $\boldsymbol{Y}$, and its response surface equation is

$$g_i=g_i(\boldsymbol{X})=f_i(\boldsymbol{Y}) \tag{A7}$$

Expand to Taylor series at the verification point $\boldsymbol{y}^*$ and take the first order term, we can get

$$g_{Li}=f_i(\boldsymbol{y}^*)+(\boldsymbol{Y}-\boldsymbol{y}^*)^{\mathrm{T}}\nabla f_i(\boldsymbol{y}^*) \tag{A8}$$

where $\nabla f_i(\boldsymbol{y}^*)$ is the gradient vector of the function $f_i(\boldsymbol{Y})$ at the checking point $\boldsymbol{y}^*$. Find the mean and variance of g_{Li}. Note that $\boldsymbol{Y}$ is a mutually independent standard normal distribution, that is, the mean of $\boldsymbol{Y}_i$ is 0 and the standard deviation is 1. Substituting into the Eq.(A8) and further finding the reliability index, there is

$$\beta_i=\frac{\mu_{g_{Li}}}{\sigma_{g_{Li}}}=-\boldsymbol{y}^{*\mathrm{T}}\frac{\nabla f_i(\boldsymbol{y}^*)}{\|\nabla f_i(\boldsymbol{y}^*)\|} \tag{A9}$$

where $\|\nabla f_i(\boldsymbol{y}^*)\|$ is the modulus of $\nabla f_i(\boldsymbol{y}^*)$, and $\nabla f_i(\boldsymbol{y}^*)/\|\nabla f_i(\boldsymbol{y}^*)\|$ is the unit vector. Therefore, $-\boldsymbol{y}^*/\beta_i=\nabla f_i(\boldsymbol{y}^*)/\|\nabla f_i(\boldsymbol{y}^*)\|$. In the same way, expand the response surface equation of another failure mode at its checking point and take the first order term to obtain

$$g_{Lj}=f_j(\boldsymbol{z}^*)+(\boldsymbol{Z}-\boldsymbol{z}^*)^{\mathrm{T}}\nabla f_j(\boldsymbol{z}^*) \tag{A10}$$

Its reliability index is

$$\beta_j=\frac{\mu_{g_{Lj}}}{\sigma_{g_{Lj}}}=-\boldsymbol{z}^{*\mathrm{T}}\frac{\nabla f_i(\boldsymbol{z}^*)}{\|\nabla f_i(\boldsymbol{z}^*)\|} \tag{A11}$$

In Eqs. (A8) and (A10), $\boldsymbol{Y}$ and $\boldsymbol{Z}$ are related. If the deviation of the correlation coefficient of random variables caused by the conversion from the original space to the standard normal space is ignored, there is

$$\mathrm{COV}(\boldsymbol{Y},\boldsymbol{Z})=\left[r_{ij}^{(kk)}\right]=\begin{bmatrix}r_{ij}^{(11)}&&&\\&r_{ij}^{(22)}&&\\&&\cdots&\\&&&r_{ij}^{(nn)}\end{bmatrix} \tag{A12}$$

Therefore, the correlation coefficient between g_{Li} and g_{Lj} is

$$\lambda_{ij}=\frac{\mathrm{COV}(g_{Li},g_{Lj})}{\sqrt{D(g_{Li})}\sqrt{D(g_{Lj})}}=\frac{\nabla f_i^{\mathrm{T}}(\boldsymbol{y}^*)\left[r_{ij}^{(kk)}\right]\nabla f_j(\boldsymbol{z}^*)}{\|\nabla f_i(\boldsymbol{y}^*)\|\times\|\nabla f_i(\boldsymbol{z}^*)\|} \tag{A13}$$

Substituting Eqs. (A9) and (A11) into Eq.(A13), we have

$$\lambda_{ij}=\frac{\nabla f_i^{\mathrm{T}}(\boldsymbol{y}^*)\left[r_{ij}^{(kk)}\right]\nabla f_j(\boldsymbol{z}^*)}{\|\nabla f_i(\boldsymbol{y}^*)\|\times\|\nabla f_i(\boldsymbol{z}^*)\|}=\frac{\boldsymbol{y}^{*\mathrm{T}}\left[r_{ij}^{(kk)}\right]\boldsymbol{z}^*}{\beta_i\beta_j} \tag{A14}$$