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Analytical solution to consolidation of unsaturated soil by vertical drains with continuous permeable boundary

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Abstract: In this paper, based on the axisymmetric consolidation theory of unsaturated soil and equal-strain assumption, an analytical solution using the homogenization of boundary conditions and eigenfunction method is proposed to three-dimensional consolidation of unsaturated soil enhanced by vertical drains under instantaneous loading, in which the continuous permeable boundary conditions are properly introduced. Then, the proposed solution is verified by the special cases of double drainage boundary conditions. Finally, the solution is analyzed using examples and the results show that the proposed solution can be used to simulate the arbitrary distribution of permeability of the top and bottom boundary by setting reasonable interface parameters, which makes up for the problem that the permeability of the top and bottom boundary is between pervious and impervious condition or follows an asymmetric distribution. In addition, with a proper ratio of influence radius to drainage well radius and appropriate depth of vertical drain, the influence of vertical flows on the dissipation of excess pore pressures is small when the ratio of radial to vertical permeability coefficient is greater than two. Last but not the least, the above influence of excess pore pressures is more obvious with the enhancement of the permeability of the top and bottom boundary considering the vertical flows.

Keywords: unsaturated soil; analytical solution; continuous permeable boundary; three-dimensional consolidation; vertical drain

1 Introduction

In the design and construction of highway and railway subgrades, methods such as surcharge preloading and surcharge preloading with vertical drains are usually used for ground treatment. The settlement of the treated ground is generally predicted using the traditional saturated soil consolidation theory. However, in the project, compacted soil, subgrade filling, shallow replacement soil and part of offshore soft soil are all unsaturated soils [1]. A large number of studies have shown that under the current requirements in the subgrade design code, the degree of saturation of some subgrade fillings is still between 65% and 87% after compaction[2], with their consolidation characteristics being significantly different from saturated soils. There is a deviation between the calculated value based on the consolidation theory of saturated soil and the measured value of the project[3]. Therefore, it is necessary to carry out further research on the consolidation characteristics of unsaturated soil ground with vertical drains. In the existing consolidation theory, the boundary of the soil layer is usually assumed to be completely impermeable or completely permeable. However, in engineering practice, the top and bottom boundaries of the ground are mostly semi-permeable boundaries. Therefore, the analytical solution to consolidation obtained under conventional boundary conditions has certain limitations.

In terms of the consolidation of saturated soil ground, based on Terzaghi's[4] saturated soil consolidation theory, Xie[5] analyzed soil consolidation under semi-permeable boundary conditions. However, the significance of semi-permeable boundary conditions is not clear. The drainage capacity cannot be expressed quantitatively either. Therefore, Mei et al. [6] proposed a continuous drainage boundary on the basis of Terzaghi's one-dimensional consolidation theory, and proved that this boundary assumption not only accords with the change of boundary permeability in engineering with time, but it is also relatively easier to obtain analytical solutions compared to the case using semi-permeable boundary assumption. Based on the above continuous drainage boundary conditions, Cai et al.[7] obtained a numerical solution for one-dimensional consolidation of saturated soil layered foundations using finite element analysis. In addition, the theory of consolidation under the continuous permeability boundary has also been extended in unsaturated soils [8–9].

Regarding the consolidation of unsaturated soil ground with vertical drains, based on the theory of unsaturated soil consolidation proposed by Fredlund et al.[10], Qin et al. [11] used Laplace transform and Bessel function to obtain the semi-analytical solution for radial consolidation of unsaturated soil ground with vertical drains under free-strain assumption. Zhou et al.[12] ultilized the differential quadrature method (DQM) to obtain a numerical solution for the axisymmetric consolidation of unsaturated soils. Subsequently, Ho et al.[13–14] obtained the analytical solution to the consolidation of unsaturated soil foundations through the separation of variables method and Laplace transform under the comprehensive consideration of vertical flow and radial flow, as well as the the analytical solution of the equal-strain consolidation for

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unsaturated soil ground with vertical drains considering the effect of smearing. In addition, Zhou et al.[15] adopted the eigenfunction method to obtain an analytical solution for the consolidation of unsaturated soil ground with vertical drains considering the effect of well resistance based on the axisymmetric consolidation theory. However, in the above studies on the consolidation of unsaturated soil ground with vertical drains, the boundary is regarded as the first type of boundary or the second type of boundary conditions. The effect of asymmetric boundary, semi-permeable boundary or more complicated boundary conditions on the consolidation are not considered for unsaturated soil ground with vertical drains.

Based on the continuous drainage boundary conditions proposed by Mei et al.[6], the top and bottom interfaces of the unsaturated soil ground with vertical drain are assumed to be continuous permeable boundaries, and an equal-strain consolidation model for the unsaturated soil ground with vertical drains is established. An analytical solution for the three-dimensional consolidation of unsaturated soil ground with vertical drains is derived by homogenizing the boundary conditions and using the eigenfunction method based on continuous permeable boundaries. The obtained solution comprehensively considers the combined flow in both the radial and vertical directions. Through the analysis of calculation examples, it can be known that under the premise of proper ratio of influence radius to drain well radius and well depth, when the ratio of radial and vertical permeability coefficient is greater than a certain value, only radial flow is considered in the consolidation analysis process. At the same time, this general solution can be applied to any combination of top and bottom boundary conditions, which not only makes up for the lack of asymmetric boundary conditions in unsaturated soil ground with vertical drains, but also degenerates the analytical solution to consolidation solution of unsaturated soil ground with vertical drain under complete permeable or complete impermeable boundary condition. The research results will provide a theoretical basis for the prediction of consolidation and settlement of the unsaturated soil ground with vertical drains.

2 Derivation of analytical solutions

2.1 Basic assumptions

(1) The air and water phases remain continuous (that is, the degree of saturation is between 15% and 90%).

(2) Pore water and soil particles are incompressible.

(3) The permeability coefficient and volume change coefficient of the soil are constant.

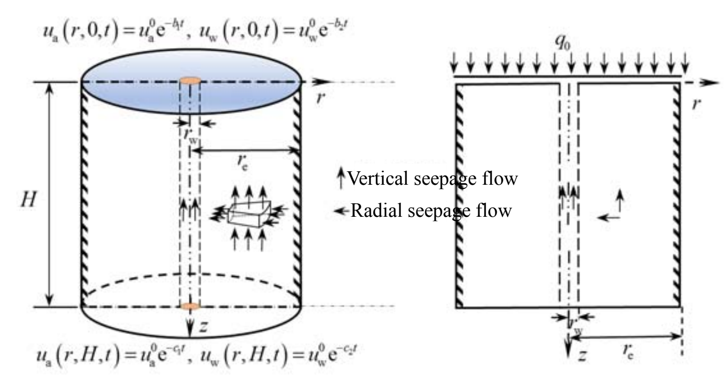
(4) The equal-strain condition is valid, that is, the vertical deformation of the soil at the same depth is equal, and there is no lateral deformation in the ground with vertical drain .

(5) The strain that occurs during consolidation is small strain.

In engineering practice, the permeability coefficient and volume change coefficient of soil usually change with time[16], but under the action of small strain and instantaneous loading, it is assumed that these parameters are constant, which has little effect on the consolidation behavior[8–15]. At the meantime, this assumption is conducive to solving the consolidation equation of unsaturated soils, and is convenient to obtain the analytical solution of the consolidation of unsaturated soils in complex situations.

2.2 Computational model

Figure 1 shows the axisymmetric consolidation model of unsaturated soil ground with vertical drians. H is the thickness of soil layer; r_w is the radius of vertical drain; r_e is the influence radius; t is the consolidation time; r and z are the radial and vertical coordinates respectively, and the range of the affected area is $r_w < r < r_e$. This paper also assumes that the top and bottom boundaries of the ground with vertical drains are continuous permeable boundaries, and the top is subjected to the vertical instantaneous uniform loading q_0; u_a^0 and u_w^0 are the initial excess pore-air pressure and the initial excess pore-water pressure, respectively; b_1 and c_1 are the air interface parameters of the top and bottom of the unsaturated soil ground with vertical drains. b_2 and c_2 are the water interface parameters of the top and bottom of the unsaturated soil ground with vertical drains.



(a) 3D schematic (b) 2D schematic

Fig. 1 Three-dimensional consolidation modeling of unsaturated soil with vertical drains

2.3 Governing equation

According to the axisymmetric consolidation theory of unsaturated soil under the assumption of equal-strain[14–15], the control equations of excess pore-air pressure and excess pore-water pressure in the consolidation process of unsaturated soil ground with vertical drains considering radial and vertical flows under instantaneous load (See Appendix A for the derivation process) can be expressed as follows:

$$\left.\begin{aligned}\frac{\partial\bar{u}_{\mathrm{a}}}{\partial t}&=-C_{\mathrm{a}}\frac{\partial\bar{u}_{\mathrm{w}}}{\partial t}-C_{\mathrm{vr}}^{\mathrm{a}}\left(\frac{\partial^2 u_{\mathrm{a}}}{\partial r^2}+\frac{1}{r}\frac{\partial u_{\mathrm{a}}}{\partial r}\right)-C_{\mathrm{vz}}^{\mathrm{a}}\frac{\partial^2\bar{u}_{\mathrm{a}}}{\partial z^2}\\ \frac{\partial\bar{u}_{\mathrm{w}}}{\partial t}&=-C_{\mathrm{w}}\frac{\partial\bar{u}_{\mathrm{a}}}{\partial t}-C_{\mathrm{vr}}^{\mathrm{w}}\left(\frac{\partial^2 u_{\mathrm{w}}}{\partial r^2}+\frac{1}{r}\frac{\partial u_{\mathrm{w}}}{\partial r}\right)-C_{\mathrm{vz}}^{\mathrm{w}}\frac{\partial^2\bar{u}_{\mathrm{w}}}{\partial z^2}\end{aligned}\right\} \tag{1}$$

where

$$\left.\begin{aligned}\bar{u}_{\mathrm{a}}&=\frac{1}{\pi\left(r_{\mathrm{e}}\right)^2-\left(r_{\mathrm{w}}\right)^2}\int_{r_{\mathrm{w}}}^{r_{\mathrm{e}}}u_{\mathrm{a}}2\pi r\mathrm{d}r\\ \bar{u}_{\mathrm{w}}&=\frac{1}{\pi\left(r_{\mathrm{e}}\right)^2-\left(r_{\mathrm{w}}\right)^2}\int_{r_{\mathrm{w}}}^{r_{\mathrm{e}}}u_{\mathrm{w}}2\pi r\mathrm{d}r\end{aligned}\right\} \tag{2}$$

$$C_{\mathrm{a}}=\frac{m_2^{\mathrm{a}}}{m_{1\mathrm{k}}^{\mathrm{a}}-m_2^{\mathrm{a}}-u_{\mathrm{atm}}(1-S_{\mathrm{r0}})n_0\Big/\left(\tilde{u}_{\mathrm{a}}^0\right)^2} \tag{3}$$

$$C_{\mathrm{vr}}^{\mathrm{a}}=\frac{k_{\mathrm{a}r}RT_{\mathrm{at}}}{g\tilde{u}_{\mathrm{a}}^0 M\left[m_{1\mathrm{k}}^{\mathrm{a}}-m_2^{\mathrm{a}}-u_{\mathrm{atm}}(1-S_{\mathrm{r0}})n_0\Big/\left(\tilde{u}_{\mathrm{a}}^0\right)^2\right]} \tag{4}$$

$$C_{\mathrm{vz}}^{\mathrm{a}}=\frac{k_{\mathrm{az}}RT_{\mathrm{at}}}{g\tilde{u}_{\mathrm{a}}^0 M\left[m_{1\mathrm{k}}^{\mathrm{a}}-m_2^{\mathrm{a}}-u_{\mathrm{atm}}(1-S_{\mathrm{r0}})n_0\Big/\left(\tilde{u}_{\mathrm{a}}^0\right)^2\right]} \tag{5}$$

$$C_{\mathrm{w}}=\frac{m_{1\mathrm{k}}^{\mathrm{w}}-m_2^{\mathrm{w}}}{m_2^{\mathrm{w}}},\ C_{\mathrm{vr}}^{\mathrm{w}}=\frac{k_{\mathrm{w}r}}{\gamma_{\mathrm{w}}m_2^{\mathrm{w}}},\ C_{\mathrm{vz}}^{\mathrm{w}}=\frac{k_{\mathrm{wz}}}{\gamma_{\mathrm{w}}m_2^{\mathrm{w}}} \tag{6}$$

where $u_{\mathrm{a}}(r,z,t)$ and $u_{\mathrm{w}}(r,z,t)$ are the excess pore-air pressure and pore-water pressure (kPa) in the affected zone, respectively; $\bar{u}_{\mathrm{a}}(z,t)$ and $\bar{u}_{\mathrm{w}}(z,t)$ are the radial average excess pore-air pressure and average excess pore-water pressure (kPa) under equal- strain conditions, respectively; $\tilde{u}_{\mathrm{a}}^0$ represents the initial value of $\tilde{u}_{\mathrm{a}}$, $\tilde{u}_{\mathrm{a}}$ is the absolute average excess pore-air pressure, $\tilde{u}_{\mathrm{a}}=\bar{u}_{\mathrm{a}}+u_{\mathrm{atm}}$, and u_{atm} is the atmospheric pressure; $m_{1\mathrm{k}}^{\mathrm{a}}$ and $m_{1\mathrm{k}}^{\mathrm{w}}$ are the volume change coefficients of air and water phases under net stress (kPa^{-1}); m_2^{a} and m_2^{w} are the volume change coefficients of air and water phases under the action of matrix suction (kPa^{-1}); $k_{\mathrm{a}r}$ (k_{az}) and $k_{\mathrm{w}r}$ (k_{wz}) refer to the permeability coefficients of air and water phases in the unsaturated soil in the radial (vertical) direction (m/s) ; M is the average air molar mass (kg/mol); R is the air constant (8.314 J/(mol·K)); T_{at} is the absolute temperature (K); S_{r0} is the initial degree of saturation; n_0 is the initial porosity; γ_{w} is the unit weight(kg/ m^3); C_{a}, $C_{\mathrm{vr}}^{\mathrm{a}}$, $C_{\mathrm{vz}}^{\mathrm{a}}$, C_{w}, $C_{\mathrm{vr}}^{\mathrm{w}}$ and $C_{\mathrm{vz}}^{\mathrm{w}}$ are all parameters; and g is the gravity acceleration (m/s^2).

2.4 Initial conditions and boundary conditions

2.4.1 Initial conditions

$$u_{\mathrm{a}}(r,z,0)=u_{\mathrm{a}}^0,\ u_{\mathrm{w}}(r,z,0)=u_{\mathrm{w}}^0 \tag{7}$$

2.4.2 Boundary conditions

At $r=r_{\mathrm{w}}$, there is

$$u_{\mathrm{a}}(r_{\mathrm{w}},z,t)=u_{\mathrm{w}}(r_{\mathrm{w}},z,t)=0 \tag{8}$$

At $r=r_{\mathrm{e}}$, there is

$$\left.\frac{\partial u_{\mathrm{a}}(r,z,t)}{\partial r}\right|_{r=r_{\mathrm{e}}}=\left.\frac{\partial u_{\mathrm{w}}(r,z,t)}{\partial r}\right|_{r=r_{\mathrm{e}}}=0 \tag{9}$$

At $z=0$, there is

$$u_{\mathrm{a}}(r,0,t)=u_{\mathrm{a}}^0\mathrm{e}^{-b_1 t},\ u_{\mathrm{w}}(r,0,t)=u_{\mathrm{w}}^0\mathrm{e}^{-b_2 t} \tag{10}$$

At $z=H$, there is

$$u_{\mathrm{a}}(r,H,t)=u_{\mathrm{a}}^0\mathrm{e}^{-c_1 t},\ u_{\mathrm{w}}(r,H,t)=u_{\mathrm{w}}^0\mathrm{e}^{-c_2 t} \tag{11}$$

The values of continuous permeable boundary interface parameters b_1, c_1, and b_2, c_2 in Eqs. (10) and (11) refer to the analysis of related parameters of unsaturated soil consolidation based on continuous boundary[8–9]. In engineering practice, pore pressure sensors can be placed at the top and bottom interfaces of the unsaturated soil ground with vertical drain to collect data and to acquire the relationship between pore pressure and time. Finally, the specific interface parameter values can be obtained through inversion methods [6, 8] .

From Eqs. (10) and (11), it can be seen that the continuous permeable boundary can strictly satisfy the initial conditions of the boundary, and it can also reflect the overall trend that the permeability of the top and bottom interfaces decrease monotonically during the consolidation process. When b_1, c_1, b_2, $c_2\to 0$, it can be degenerated into the impermeable top and impermeable bottom (ITIB) condition; when b_1, c_1, b_2, $c_2\to\infty$, it can be degenerated into the permeable top and permeable bottom (PTPB) condition. When the appropriate interface parameters are selected, the boundary conditions with asymmetric permeability can also be obtained, for instance, the permeable top and impermeable bottom (PTIB) condition, the semipermeable top and ipermeable bottom (STIB) condition , the semipermeable top and semipermeable bottom (STSB) condition but different in permeability, etc.

2.5 Derivation of analytical solution

Reorganize the governing Eqs. (1) into

$$\left.\begin{aligned}\frac{\partial}{\partial r}\left(r\frac{\partial u_{\mathrm{a}}}{\partial r}\right)&=\frac{r}{C_{\mathrm{vr}}^{\mathrm{a}}}\left(-C_{\mathrm{a}}\frac{\partial\bar{u}_{\mathrm{w}}}{\partial t}-\frac{\partial\bar{u}_{\mathrm{a}}}{\partial t}-C_{\mathrm{vz}}^{\mathrm{a}}\frac{\partial^2\bar{u}_{\mathrm{a}}}{\partial z^2}\right)\\ \frac{\partial}{\partial r}\left(r\frac{\partial u_{\mathrm{w}}}{\partial r}\right)&=\frac{r}{C_{\mathrm{vr}}^{\mathrm{w}}}\left(-C_{\mathrm{w}}\frac{\partial\bar{u}_{\mathrm{a}}}{\partial t}-\frac{\partial\bar{u}_{\mathrm{w}}}{\partial t}-C_{\mathrm{vz}}^{\mathrm{w}}\frac{\partial^2\bar{u}_{\mathrm{w}}}{\partial z^2}\right)\end{aligned}\right\} \tag{12}$$

After Eqs. (12) is integrated on r and combined with Eqs. (8) and (9) of the boundary conditions, it is substituted into the Eqs. (2) to obtain

$$\left.\begin{aligned}\bar{u}_{\mathrm{a}}&=A\left(C_{\mathrm{a}}\frac{\partial\bar{u}_{\mathrm{w}}}{\partial t}+\frac{\partial\bar{u}_{\mathrm{a}}}{\partial t}+C_{\mathrm{vz}}^{\mathrm{a}}\frac{\partial^2\bar{u}_{\mathrm{a}}}{\partial z^2}\right)\\ \bar{u}_{\mathrm{w}}&=W\left(C_{\mathrm{w}}\frac{\partial\bar{u}_{\mathrm{a}}}{\partial t}+\frac{\partial\bar{u}_{\mathrm{w}}}{\partial t}+C_{\mathrm{vz}}^{\mathrm{w}}\frac{\partial^2\bar{u}_{\mathrm{w}}}{\partial z^2}\right)\end{aligned}\right\} \tag{13}$$

where

$$\left.\begin{aligned}&A=\frac{\left(r_{\mathrm{e}}\right)^2F}{2C_{\mathrm{vr}}^{\mathrm{a}}},\ W=\frac{\left(r_{\mathrm{e}}\right)^2F}{2C_{\mathrm{vr}}^{\mathrm{w}}},\ N=\frac{r_{\mathrm{e}}}{r_{\mathrm{w}}}\\ &F=\frac{N^2}{N^2-1}\left(\ln N+\frac{1}{N^2}-\frac{1}{4N^4}-\frac{3}{4}\right)\end{aligned}\right\} \tag{14}$$

Equations (13) are the homogeneous partial differential equations, and their boundary conditions Eqs. (10) and (11) are non-homogeneous boundary conditions. To solve the problem, the non-homogeneous boundary conditions can be homogenized, that is, let

$$\left.\begin{aligned}\bar{u}_{\mathrm{a}}&=v_{\mathrm{a}}+\frac{H-z}{H}u_{\mathrm{a}}^{0}\mathrm{e}^{-b_1t}+\frac{z}{H}u_{\mathrm{a}}^{0}\mathrm{e}^{-c_1t}\\ \bar{u}_{\mathrm{w}}&=v_{\mathrm{w}}+\frac{H-z}{H}u_{\mathrm{w}}^{0}\mathrm{e}^{-b_2t}+\frac{z}{H}u_{\mathrm{w}}^{0}\mathrm{e}^{-c_2t}\end{aligned}\right\}\tag{15}$$

At this moment, $v_{\mathrm{a}}(z, t)$ and $v_{\mathrm{w}}(z, t)$ are intermediate variables, and their boundary and initial conditions are: $v_{\mathrm{a}}(0, t) = v_{\mathrm{w}}(0, t) = 0$, $v_{\mathrm{a}}(H, t) = v_{\mathrm{w}}(H, t) = 0$; $v_{\mathrm{a}}(z, 0) = v_{\mathrm{w}}(z, 0) = 0$.

Substituting Eqs. (15) into Eqs. (13), then

$$\left.\begin{aligned}&\frac{\partial v_{\mathrm{a}}}{\partial t}-\frac{1}{A}v_{\mathrm{a}}+C_{\mathrm{vz}}^{\mathrm{a}}\frac{\partial^2 v_{\mathrm{a}}}{\partial z^2}+C_{\mathrm{a}}\frac{\partial v_{\mathrm{w}}}{\partial t}=\varphi\\ &\frac{\partial v_{\mathrm{w}}}{\partial t}-\frac{1}{W}v_{\mathrm{w}}+C_{\mathrm{vz}}^{\mathrm{w}}\frac{\partial^2 v_{\mathrm{w}}}{\partial z^2}+C_{\mathrm{w}}\frac{\partial v_{\mathrm{a}}}{\partial t}=\psi\end{aligned}\right\}\tag{16}$$

where

$$\begin{aligned}\varphi(z,t)=&\frac{H-z}{H}\left(b_1u_{\mathrm{a}}^{0}\mathrm{e}^{-b_1t}+C_{\mathrm{a}}b_2u_{\mathrm{w}}^{0}\mathrm{e}^{-b_2t}+\frac{1}{A}u_{\mathrm{a}}^{0}\mathrm{e}^{-b_1t}\right)+\\ &\frac{z}{H}\left(c_1u_{\mathrm{a}}^{0}\mathrm{e}^{-c_1t}+C_{\mathrm{a}}c_2u_{\mathrm{w}}^{0}\mathrm{e}^{-c_2t}+\frac{1}{A}u_{\mathrm{a}}^{0}\mathrm{e}^{-c_1t}\right)\end{aligned}\tag{17}$$

$$\begin{aligned}\psi(z,t)=&\frac{H-z}{H}\left(b_2u_{\mathrm{w}}^{0}\mathrm{e}^{-b_2t}+C_{\mathrm{w}}b_1u_{\mathrm{a}}^{0}\mathrm{e}^{-b_1t}+\frac{1}{W}u_{\mathrm{w}}^{0}\mathrm{e}^{-b_2t}\right)+\\ &\frac{z}{H}\left(c_2u_{\mathrm{w}}^{0}\mathrm{e}^{-c_2t}+C_{\mathrm{w}}c_1u_{\mathrm{a}}^{0}\mathrm{e}^{-c_1t}+\frac{1}{W}u_{\mathrm{w}}^{0}\mathrm{e}^{-c_2t}\right)\end{aligned}\tag{18}$$

Expand $v_{\mathrm{a}}(z,t)$, $v_{\mathrm{w}}(z,t)$ according to eigenfunction:

$$\left.\begin{aligned}v_{\mathrm{a}}(z,t)&=\sum_{j=1}^{\infty}\omega_j(t)\sin\frac{J}{H}z\\ v_{\mathrm{w}}(z,t)&=\sum_{j=1}^{\infty}\xi_j(t)\sin\frac{J}{H}z\end{aligned}\right\}\tag{19}$$

where $J = j\pi$, ($j = 1,2,3,\cdots$); $\omega_j(0)=0$, $\xi_j(0)=0$.

Substituting Eqs. (19) into Eqs. (16), according to the orthogonality of the trigonometric function sin(Jz/H) in the interval [0, H], and so

$$\left.\begin{aligned}\omega_j(t)&=\lambda_{\mathrm{w}}\frac{\mathrm{d}\xi_j(t)}{\mathrm{d}t}+\lambda_{\mathrm{a}}\frac{\mathrm{d}\omega_j(t)}{\mathrm{d}t}-\lambda_{\mathrm{q}}\varphi_j(t)\\ \xi_j(t)&=\eta_{\mathrm{a}}\frac{\mathrm{d}\omega_j(t)}{\mathrm{d}t}+\eta_{\mathrm{w}}\frac{\mathrm{d}\xi_j(t)}{\mathrm{d}t}-\eta_{\mathrm{q}}\psi_j(t)\end{aligned}\right\}\tag{20}$$

where

$$\lambda_{\mathrm{a}}=\frac{AH^2}{H^2+AC_{\mathrm{vz}}^{\mathrm{a}}J^2},\ \eta_{\mathrm{a}}=\frac{WC_{\mathrm{w}}H^2}{H^2+WC_{\mathrm{vz}}^{\mathrm{w}}J^2}\tag{21}$$

$$\lambda_{\mathrm{w}}=\frac{AC_{\mathrm{a}}H^2}{H^2+AC_{\mathrm{vz}}^{\mathrm{a}}J^2},\ \eta_{\mathrm{w}}=\frac{WH^2}{H^2+WC_{\mathrm{vz}}^{\mathrm{w}}J^2}\tag{22}$$

$$\lambda_{\mathrm{q}}=\frac{2AH^2}{H^2J+AC_{\mathrm{vz}}^{\mathrm{a}}J^3},\ \eta_{\mathrm{q}}=\frac{2WH^2}{H^2J+WC_{\mathrm{vz}}^{\mathrm{w}}J^3}\tag{23}$$

$$\begin{aligned}\varphi_j(t)=&\left(b_1+\frac{1}{A}\right)u_{\mathrm{a}}^{0}\mathrm{e}^{-b_1t}+C_{\mathrm{a}}b_2u_{\mathrm{w}}^{0}\mathrm{e}^{-b_2t}-\\ &(-1)^j\left(c_1+\frac{1}{A}\right)u_{\mathrm{a}}^{0}\mathrm{e}^{-c_1t}-(-1)^jC_{\mathrm{a}}c_2u_{\mathrm{w}}^{0}\mathrm{e}^{-c_2t}\end{aligned}\tag{24}$$

$$\begin{aligned}\psi_j(t)=&C_{\mathrm{w}}b_1u_{\mathrm{a}}^{0}\mathrm{e}^{-b_1t}+\left(b_2+\frac{1}{W}\right)u_{\mathrm{w}}^{0}\mathrm{e}^{-b_2t}-\\ &(-1)^jC_{\mathrm{w}}c_1u_{\mathrm{a}}^{0}\mathrm{e}^{-c_1t}-(-1)^j\left(c_2+\frac{1}{W}\right)u_{\mathrm{w}}^{0}\mathrm{e}^{-c_2t}\end{aligned}\tag{25}$$

Introducing arbitrary constants q_1 and q_2, considering Eqs. (20), and so

$$\begin{aligned}&q_1\omega_j(t)+q_2\xi_j(t)=(q_1\lambda_{\mathrm{a}}+q_2\eta_{\mathrm{a}})\frac{\mathrm{d}\omega_j(t)}{\mathrm{d}t}+\\ &(q_1\lambda_{\mathrm{w}}+q_2\eta_{\mathrm{w}})\frac{\mathrm{d}\xi_j(t)}{\mathrm{d}t}-\left[q_1\lambda_{\mathrm{q}}\varphi_j(t)+q_2\eta_{\mathrm{q}}\psi_j(t)\right]\end{aligned}\tag{26}$$

Introducing the constant Q through the variable $\Phi(t)=q_1\omega_j(t)+q_2\xi_j(t)$, from Eq. (26), and thus

$$Qq_1=q_1\lambda_{\mathrm{a}}+q_2\eta_{\mathrm{a}},\ Qq_2=q_1\lambda_{\mathrm{w}}+q_2\eta_{\mathrm{w}}\tag{27}$$

In addition, the condition for Eq. (27) to be valid is

$$(\lambda_{\mathrm{a}}-Q)(\eta_{\mathrm{w}}-Q)-\lambda_{\mathrm{w}}\eta_{\mathrm{a}}=0\tag{28}$$

The roots Q_1 and Q_2 of Eqs.(28) can be expressed as

$$Q_{1,2}=\frac{1}{2}\left[\lambda_{\mathrm{a}}+\eta_{\mathrm{w}}\mp\sqrt{(\lambda_{\mathrm{a}}-\eta_{\mathrm{w}})^2+4\lambda_{\mathrm{w}}\eta_{\mathrm{a}}}\right]\tag{29}$$

When $Q=Q_1$, q_1 and q_2 in Eq. (27) are q_{11} and q_{21}, respectively. Correspondingly, when $Q=Q_2$, q_1 and q_2 are q_{12} and q_{22}, respectively.

In summary, Eq. (26) can be expressed as the following first-order linear non-homogeneous ordinary differential equations:

$$\left.\begin{aligned}\Phi_1(t)&=Q_1\frac{\mathrm{d}\Phi_1(t)}{\mathrm{d}t}-\beta_1(t)\\ \Phi_2(t)&=Q_2\frac{\mathrm{d}\Phi_2(t)}{\mathrm{d}t}-\beta_2(t)\end{aligned}\right\}\tag{30}$$

where

$$\left.\begin{aligned}&\Phi_1(t)=\omega_j(t)+q_{21}\xi_j(t),\ \Phi_2(t)=q_{12}\omega_j(t)+\xi_j(t)\\ &q_{12}=\eta_{\mathrm{a}}/(Q_2-\lambda_{\mathrm{a}}),\ \beta_1(t)=\lambda_{\mathrm{q}}\varphi_j(t)+q_{21}\eta_{\mathrm{q}}\psi_j(t)\\ &q_{21}=\lambda_{\mathrm{w}}/(Q_1-\eta_{\mathrm{w}}),\ \beta_2(t)=q_{12}\lambda_{\mathrm{q}}\varphi_j(t)+\eta_{\mathrm{q}}\psi_j(t)\end{aligned}\right\}\tag{31}$$

The general solution of Eqs. (30) are

$$\left.\begin{aligned}\Phi_1&=\lambda_{\mathrm{q}}\zeta_1+q_{21}\eta_{\mathrm{q}}\tau_1\\ \Phi_2&=q_{12}\lambda_{\mathrm{q}}\zeta_2+\eta_{\mathrm{q}}\tau_2\end{aligned}\right\}\tag{32}$$

where

$$\begin{aligned}\zeta_1=&C_{\mathrm{a}}u_{\mathrm{w}}^{0}\left[b_2E(Q_1,b_2)-(-1)^jc_2E(Q_1,c_2)\right]+\\ &u_{\mathrm{a}}^{0}\left[(b_1+A^{-1})E(Q_1,b_1)-(-1)^j(c_1+A^{-1})E(Q_1,c_1)\right]\end{aligned}\tag{33}$$

$$\zeta_2 = C_a u_w^0 \left[b_2 E(Q_2, b_2) - (-1)^j c_2 E(Q_2, c_2) \right] + u_a^0 \left[\left(b_1 + A^{-1}\right) E(Q_2, b_1) - (-1)^j \left(c_1 + A^{-1}\right) E(Q_2, c_1) \right] \tag{34}$$

$$\tau_1 = C_w u_a^0 \left[b_1 E(Q_1, b_1) - (-1)^j c_1 E(Q_1, c_1) \right] + u_w^0 \left[\left(b_2 + W^{-1}\right) E(Q_1, b_2) - (-1)^j \left(c_2 + W^{-1}\right) E(Q_1, c_2) \right] \tag{35}$$

$$\tau_2 = C_w u_a^0 \left[b_1 E(Q_2, b_1) - (-1)^j c_1 E(Q_2, c_1) \right] + u_w^0 \left[\left(b_2 + W^{-1}\right) E(Q_2, b_2) - (-1)^j \left(c_2 + W^{-1}\right) E(Q_2, c_2) \right] \tag{36}$$

$$E(Q, D) = \frac{e^{t/Q} - e^{-Dt}}{1 + DQ}, \; Q \in (Q_1, Q_2), \; D \in (b_1, b_2, c_1, c_2) \tag{37}$$

Combining Eqs. (15), (19) and (32) results in:

$$\left.\begin{aligned} \bar{u}_a &= \sum_{j=1}^{\infty} \omega_j(t) \sin\frac{J}{H} z + \frac{H - z}{H} u_a^0 e^{-b_1 t} + \frac{z}{H} u_a^0 e^{-c_1 t} \\ \bar{u}_w &= \sum_{j=1}^{\infty} \xi_j(t) \sin\frac{J}{H} z + \frac{H - z}{H} u_w^0 e^{-b_2 t} + \frac{z}{H} u_w^0 e^{-c_2 t} \end{aligned}\right\} \tag{38}$$

where

$$\omega_j(t) = \frac{q_{21}\Phi_2 - \Phi_1}{q_{12} q_{21} - 1}, \; \xi_j(t) = \frac{q_{12}\Phi_1 - \Phi_2}{q_{12} q_{21} - 1} \tag{39}$$

Equations (38) are the analytical solution to equal-strain consolidation of axisymmetric unsaturated soil ground with vertical drain under instantaneous uniform loading considering continuous permeable seepage boundary conditions.

3 Verification and example analysis

An example is used to verify the analytical solution of the axisymmetric consolidation of unsaturated soils with vertical drains under the continuous permeable boundary. And the consolidation characteristics of the ground is also analyzed. In the example used in this paper, the physical parameters are consistent with those adopted by Qin et al. [11], and the parameters are as follows: $k_{wr} = 10^{-10}$ m/s, $k_{wr} = 2k_{wz}$, $S_{r0} = 80\%$, $n_0 = 50\%$, $r_w = 0.2$ m, $r_e = 1.8$ m, $H = 5$ m, $m_{1k}^w = -5 \times 10^{-5}$ kPa^{-1}, $m_{1k}^a = -2 \times 10^{-4}$ kPa^{-1}, $m_2^w = -2 \times 10^{-4}$ kPa^{-1}, $m_2^a = 1 \times 10^{-4}$ kPa^{-1}, $q_0 = 100$ kPa, $u_a^0 = 20$ kPa, $u_w^0 = 40$ kPa。

3.1 Verification

When b_1, c_1, b_2, $c_2 \to 0$, the analytical solution in this paper can be degenerated into an analytical solution of equal-strain consolidation of unsaturated soil ground with vertical drains under the ITIB condition, and so

$$\left.\begin{aligned} \bar{u}_a &= \sum_{j=1}^{\infty} \left[1 - (-1)^j\right] \Omega_1(t) \sin\frac{J}{H} z + u_a^0 \\ \bar{u}_w &= \sum_{j=1}^{\infty} \left[1 - (-1)^j\right] \mho_1(t) \sin\frac{J}{H} z + u_w^0 \end{aligned}\right\} \tag{40}$$

where

$$\Omega_1(t) = \frac{q_{21}\theta_2 - \theta_1}{q_{12} q_{21} - 1}, \mho_1(t) = \frac{q_{12}\theta_1 - \theta_2}{q_{12} q_{21} - 1} \tag{41}$$

$$\theta_1 = \left(e^{t/Q_1} - 1\right)\left(\lambda_q A^{-1} u_a^0 + q_{21} \eta_q W^{-1} u_w^0\right) \tag{42}$$

$$\theta_2 = \left(e^{t/Q_2} - 1\right)\left(q_{12} \lambda_q A^{-1} u_a^0 + \eta_q W^{-1} u_w^0\right) \tag{43}$$

However, when b_1, c_1, b_2, $c_2 \to \infty$, the analytical solution of equal-strain consolidation under continuous permeable boundary conditions in this paper is the analytical solution of equal-strain consolidation of unsaturated soil ground with vertical drain under the PTPB condition:

$$\left.\begin{aligned} \bar{u}_a &= \sum_{j=1}^{\infty} \left[1 - (-1)^j\right] \Omega_2(t) \sin\frac{J}{H} z \\ \bar{u}_w &= \sum_{j=1}^{\infty} \left[1 - (-1)^j\right] \mho_2(t) \sin\frac{J}{H} z \end{aligned}\right\} \tag{44}$$

where

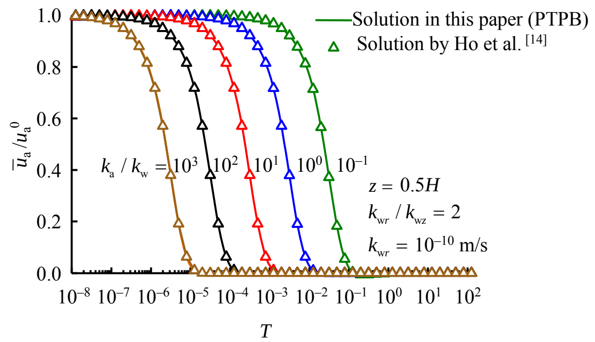
$$\Omega_2(t) = \frac{q_{21}\phi_2 - \phi_1}{q_{12} q_{21} - 1}, \mho_2(t) = \frac{q_{12}\phi_1 - \phi_2}{q_{12} q_{21} - 1} \tag{45}$$

$$\phi_1 = \left[\lambda_q \left(u_a^0 + C_a u_w^0\right) + q_{21} \eta_q \left(C_w u_a^0 + u_w^0\right)\right] (Q_1)^{-1} e^{t/Q_1} \tag{46}$$

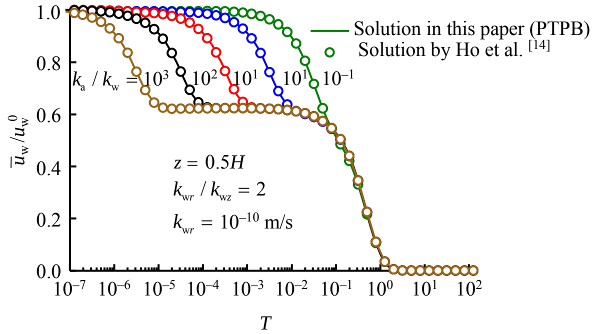
$$\phi_2 = \left[q_{12} \lambda_q \left(u_a^0 + C_a u_w^0\right) + \eta_q \left(C_w u_a^0 + u_w^0\right)\right] (Q_2)^{-1} e^{t/Q_2} \tag{47}$$

The correctness of the analytical solution in this paper is to verified by comparing Eqs.(44) with the result of Ho et al.[14]

At z = 0.5H, the comparison curves that $\bar{u}_a$ and $\bar{u}_w$ dissipate with time under the PTPB condition is plotted in Fig.2, where the time factor $T = k_{wr} t / \left(\gamma_w m_{1k}^s r_e^2\right)$. The dissipation curves in Fig. 2(b) are clearly divided into two stages when k_a / k_w is greater than 1, and there is a plateau in the middle of the two stages. This may be due to the fact that $\bar{u}_w$ needs an adjustment period to continue to dissipate due to the existence of suction just after the dissipation of $\bar{u}_a$. Comparison between Figs. 2(a) and 2(b) demonstrates that when the excess pore-air pressure dissipates to 0, it is just the beginning of the plateau, which is also the diving point between the early and late periods of the dissipation curve. But the plateaus are at the same position when different values of k_a / k_w are taken. This is because when the initial degree of saturation and other parameters are the same, only changing the air permeability coefficient will only affect the rate of dissipation of $\bar{u}_a$, but will not affect the degree of dissipation of $\bar{u}_w$. In addition, after the end of the dissipation of $\bar{u}_a$, the onset time of the dissipation of $\bar{u}_w$ and the dissipation rate tend to be consistent due to the same values of the water phase related parameters. The above phenomenon is almost consistent with the results of Ho et al.[14] when ignoring the smearing effect, implying that the analytical solution in this paper is reliable.



(a) $\bar{u}_a / u_a^0 - T$ curves



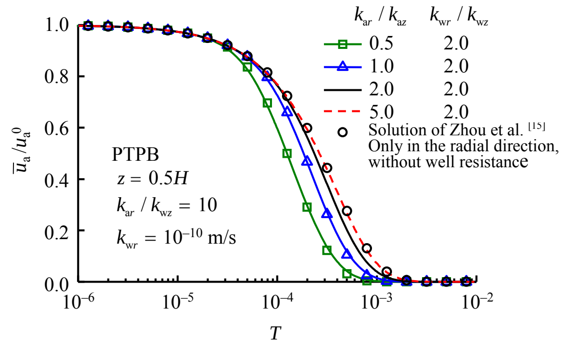
(b) $\bar{u}_w / u_w^0 - T$ curves

Fig. 2 Comparison of the solution in this paper under PTPB condition with the result of Ho et al. [14]

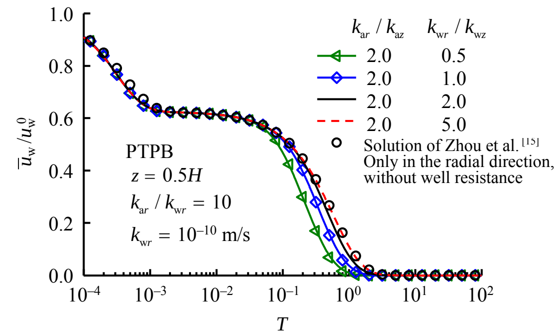
3.2 Example analysis

To explore the three-dimensional consolidation characteristics of unsaturated soil ground with vertical drains under continuous permeable boundary conditions, in this paper, calculation examples are used to compare the excess pore-air pressure and excess pore-water pressure with respect to radial, vertical permeability coefficient ratios (i.e. k_{ar} / k_{az} and k_{wr} / k_{wz}), depth z, and permeability of the top and bottom boundaries of the ground with vertical drains (i.e. different values of interface parameters b_1, c_1, b_2, and c_2). The focus is on the influence of these parameters on the law of pressure dissipation.

Figure 3 shows the dissipation curves of $\bar{u}_a$ and $\bar{u}_w$ over time at z = 0.5H, with different radial and vertical permeability coefficient ratios under the PTPB conditions. The dissipation curves are compared with the existing consolidation analytical solution that only considers radial flow. It can be seen from Fig. 3 that the three-dimensional consolidation that considers both radial and vertical flows and the axisymmetric consolidation that only considers the radial flow shows that regardless of the initial excess pore-air pressure dissipation or the excess pore-water pressure dissipation, when the permeability coefficient ratio between radial and vertical flows (i.e. k_r / k_z) takes a value of 5.0, the vertical seepage has almost no effect on the dissipation of the excess pore pressures; when the k_r / k_z value is 2.0, the vertical flow has a slight effect on the dissipation of the excess pore pressures. It can be concluded that in the unsaturated soil ground with vertical drain at proper ratio of influence radius to drain well radius and well depth, when the ratio of radial to vertical permeability coefficient is greater than 2.0, the vertical flow has hardly any effect on the dissipation of excess pore pressure. Compared with the dissipation curve of $\bar{u}_a$ in Fig. 3(a), the dissipation curve of $\bar{u}_w$ in Fig. 3(b) is different. The dissipation is divided into two stages. When k_{wr} / k_{wz} ($k_{ar} / k_{az} = 2$ remains constant) takes different values, the dissipation curves in the first stage (before the end of $\bar{u}_a$ dissipation) almost coincide; until the air phase dissipation is completed and adjusted to enter the second stage, the dissipation curves are different. This indicates that in the process of three-dimensional consolidation of unsaturated soil ground with vertical drains, the first and second stages of dissipation of $\bar{u}_w$ are controlled by the dissipation of excess pore-air pressure and excess pore-water pressure, respecttively.



(a) $\bar{u}_a / u_a^0 - T$ curves

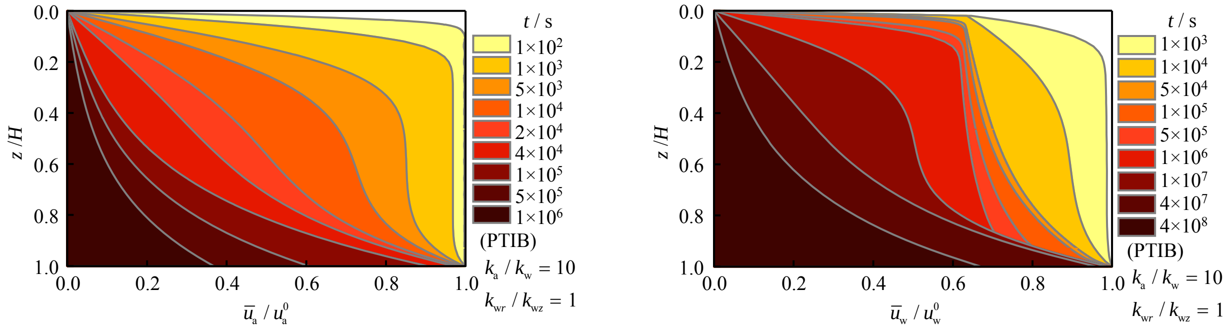


(b) $\bar{u}_w / u_w^0 - T$ curves

Fig. 3 Dissipation of average excess pore-air pressure and pore-water pressure under different ratios of radial to vertical permeability coefficients

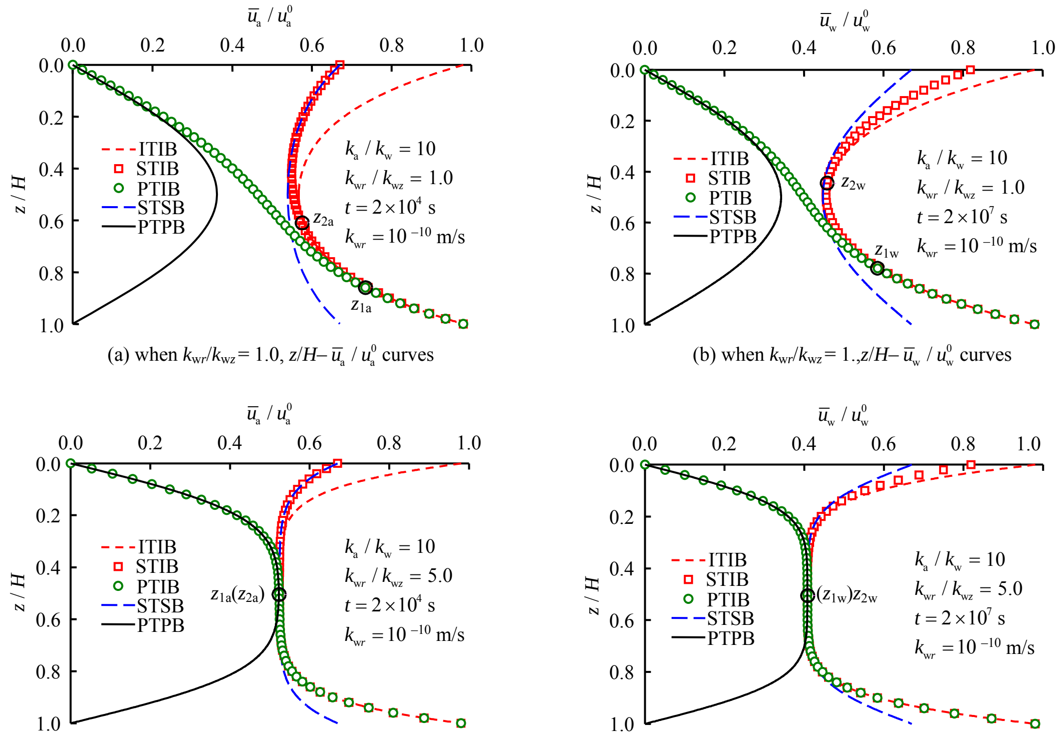
The dissipation curves of the average excess pore-air pressure and the average excess pore-water pressure over time along the depth direction is plotted in Fig. 4 in the equal-strain consolidation. The corresponding condition in Fig. 4 is the unsaturated soil ground with vertical drain under the assumption that the top boundary is completely permeable and the bottom boundary is completely impermeable (asymmetric permeable boundary of PTIB condition). Here, the interface parameters b_1, c_1, b_2, c_2 take the value of 10^2, 10^{-2}, 10^{-6}, 10^{-9} s^{-1}, respectively. It can be found that the excess pore pressures quickly dissipates to zero in the area near the completely permeable boundary; while in the area surrounding the completely impermeable boundary, the excess pore pressures remains the same as the initial pore pressure for a long period of time. By observing Fig. 4(b), it can be seen that during the period from 5×10^4 s to 10^6 s, the average excess pore-water pressure dissipation curve is close to parallel along the depth, and the average excess pore-water pressure value hardly changes during this period. This phenomenon is in line with the phenomenon of "plateau" in the dissipation curve when $k_a / k_w = 10$ at $z = 0.5H$ in Fig. 2(b).

Figure 5 shows the distributions of the average excess pore-air pressure and average excess pore-water pressure along with the depth during the consolidation process when t=2×10^4 s and t = 2×10^7s, respectively for the unsaturated soil ground with vertical drain under different interface parameters of the top and bottom boundaries. It can be observed that the –top and bottom boundary of the unsaturated soil ground with vertical drains change from completely impermeable to completely permeable with the increase of the interface parameters' values (i.e. b_1, c_1,b_2, c_2). Observing the pressure distribution curves under the assumptions of ITIB, STIB and PTIB in Figs. 5(a) and 5(b) (where ITIB: $b_1 = c_1 = 10^{-6}$ s^{-1}, $b_2 = c_2 = 10^{-9}$ s^{-1}; PTPB: $b_1 = c_1 = 10^2$ s^{-1}, $b_2 = c_2 = 10^{-2}$ s^{-1}; STSB: $b_1 = c_1 = 2\times10^{-5}$ s^{-1}, $b_2 = c_2 = 2\times10^{-8}$ s^{-1}; STIB: $b_1 = 2\times10^{-5}$ s^{-1}, $b_2 = 10^{-8}$ s^{-1}, $c_1 = 10^{-6}$ s^{-1}; $c_2 =10^{-9}$ s^{-1}; PTIB: $b_1 = 10^2$ s^{-1}, $b_2 = 10^{-2}$ s^{-1}, $c_1 = 10^{-6}$ s^{-1}; $c_2 =10^{-9}$ s^{-1}), it can be found that when the top–boundary permeability assumption changes from completely impermeable to completely permeable, the excess pore pressure dissipation curves differ at depths less than z_1; when converting from completely impermeable to semi-permeable, the difference between the two curves is



(a) z/H–$\bar{u}_a / u_a^0$ curves (b) z/H–$\bar{u}_w / u_w^0$ curves

Fig. 4 Dissipation of (a) average excess pore-air pressure $\bar{u}_a$ and (b) pore-water pressure $\bar{u}_w$ along the vertical direction with time



(a) when k_{wr}/k_{wz} = 1.0, z/H–$\bar{u}_a / u_a^0$ curves (b) when k_{wr}/k_{wz} = 1.,z/H–$\bar{u}_w / u_w^0$ curves

(c) when k_{wr}/k_{wz} = 5.0,z/H–$\bar{u}_a / u_a^0$ curves (d) when k_{wr}/k_{wz} = 5.0, z/H–$\bar{u}_w / u_w^0$ curves

Fig. 5 Distribution of average excess pore-air pressure $\bar{u}_a$ and pore-water pressure $\bar{u}_w$ along vertical direction under different interface parameters and the ratio of radial to vertical permeability coefficients

only observed when the depth is less than z_2 . In Figs. 5(c) and 5(d), when k_{wr}/k_{wz} = 5 (according to the analysis of Fig. 3, the vertical flow can be ignored at this time), $z_1 = z_2$ and the value is equal to 0.5*H*. When k_{wr}/k_{wz} is taken a different value compared to the one mentioned above, the comparison of excess pore pressure distribution shows that when the ratio of radial to vertical permeability coefficients is less than a certain value, the effect of vertical flow on the dissipation of excess pore pressures of unsaturated soil ground with vertical drain is more significant. From Figs. 5(a) and 5(b) where $z_1 > z_2$, it can be found that the larger the permeability of the top interface, the greater the effect of vertical flow on the dissipation of excess pore pressures in the depth direction. In addition, by comparing the pressure dissipation curves under the STSB and STIB conditions, it is found that only the average excess pore-water pressure changes when the top boundary interface parameter b_2 is changed.

4 Conclusion

In this paper, an analytical solution of axisymmetric consolidation for unsaturated soils ground with vertical drains based on continuous permeable boundary under instantaneous loading. Through the analysis of examples, conclusions obtained are summarized as follows:

(1) The analytical solution for the consolidation of the unsaturated soil ground with vertical drains obtained in this paper can be used to simulate the arbitrary distribution of the top and bottom boundary permeabilities in reality by setting reasonable interface parameters, which makes up for the current inability to account for the permeability at the top and bottomboundaries between permeable and impermeable, and the inability to consider the asymmetric distribution of permeability. This means that the solution in this paper is continuous and asymmetric.

(2) Under the premise of proper ratio of influence radius to drain well radius and well depth, the vertical flow has less influence on the dissipation of excess pore pressures when the ratio of radial to vertical permeability coefficients is greater than 2.

(3) When considering vertical flow, the larger the permeability at the top and bottom boundaries, the greater the influence of vertical flow on the dissipation of excess pore pressures.

(4) In different unsaturated soil ground with vertical drains, the water permeability (or air permeability) of the top and bottom interface increases with the increase of the interface parameter value, which will only accelerate the dissipation of corresponding excess pore-water pressure (or excess pore-air pressure).

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Appendix A

A1 Governing equation of water phase

For the unsaturated soil unit in the foundation of a three-dimensional axisymmetric shaft, the direction of water flowing from the foundation to the shaft in the radial direction is defined as the positive direction; the direction of water flowing vertically upward to the top surface is defined as the positive direction. The velocity of the water flowing into the unit in the radial and vertical directions are $v_{wr}+\frac{\partial v_{wr}}{\partial r}\mathrm{d}r$ and $v_{wz}+\frac{\partial v_{wz}}{\partial z}\mathrm{d}z$ respectively. Using the net flow rate of water of the unit, that is, the change in water volume is equal to the difference in the volume of water flowing into and out of the unit within a certain period of time:

$$\frac{\partial V_{\mathrm{w}}}{\partial t}=-\left(\frac{\partial v_{\mathrm{wr}}}{\partial r}+\frac{\partial v_{\mathrm{wz}}}{\partial z}\right)r\mathrm{d}r\mathrm{d}z\mathrm{d}\theta=-\left(\frac{\partial v_{\mathrm{wr}}}{\partial r}+\frac{\partial v_{\mathrm{wz}}}{\partial z}\right)V_0 \quad \text{(A1)}$$

Where V_{W} is the volume of water, $V_0=r\mathrm{d}r\mathrm{d}z\mathrm{d}\theta$.

Rearranging Eq. (A1), we can get

$$\frac{\partial\left(V_{\mathrm{w}}/V_0\right)}{\partial t}=-\left(\frac{\partial v_{\mathrm{wr}}}{\partial r}+\frac{\partial v_{\mathrm{wz}}}{\partial z}\right) \quad \text{(A2)}$$

Assuming that water seepage conforms to Darcy's law, namely

$$v_{\mathrm{wr}}=-k_{\mathrm{wr}}\frac{\partial(u_{\mathrm{w}}/\gamma_{\mathrm{w}})}{\partial r}=-\frac{k_{\mathrm{wr}}}{\gamma_{\mathrm{w}}}\frac{\partial u_{\mathrm{w}}}{\partial r} \quad \text{(A3)}$$

$$v_{\mathrm{wz}}=-k_{\mathrm{wz}}\frac{\partial(u_{\mathrm{w}}/\gamma_{\mathrm{w}})}{\partial z}=-\frac{k_{\mathrm{wz}}}{\gamma_{\mathrm{w}}}\frac{\partial u_{\mathrm{w}}}{\partial z} \quad \text{(A4)}$$

After differentiating Eqs. (A3) and (A4) with respect to *r* and *z*, respectively, substituting them into Eq. (A2), we can get

$$\frac{\partial\left(V_{\mathrm{w}}/V_0\right)}{\partial t}=\frac{k_{\mathrm{wr}}}{\gamma_{\mathrm{w}}}\left(\frac{\partial^2u_{\mathrm{w}}}{\partial r^2}+\frac{1}{r}\frac{\partial u_{\mathrm{w}}}{\partial r}\right)+\frac{k_{\mathrm{wz}}}{\gamma_{\mathrm{w}}}\frac{\partial^2u_{\mathrm{w}}}{\partial z^2} \quad \text{(A5)}$$

According to Fredlund's unsaturated soil consolidation theory and equal-strain assumption, under loading K_0:

$$\mathrm{d}V_{\mathrm{w}}/V_0=m_{1\mathrm{k}}^{\mathrm{w}}\mathrm{d}(q_0-\bar{u}_{\mathrm{a}})+m_2^{\mathrm{w}}\mathrm{d}\left(\bar{u}_{\mathrm{a}}-\bar{u}_{\mathrm{w}}\right) \quad \text{(A6)}$$

Substituting Eq.(A6) into Eq. (A5), we can obtain the governing equation for water phase:

$$\frac{\partial\bar{u}_{\mathrm{w}}}{\partial t}=-C_{\mathrm{w}}\frac{\partial\bar{u}_{\mathrm{a}}}{\partial t}-C_{\mathrm{v}r}^{\mathrm{w}}\left(\frac{\partial^2u_{\mathrm{w}}}{\partial r^2}+\frac{1}{r}\frac{\partial u_{\mathrm{w}}}{\partial r}\right)-C_{\mathrm{vz}}^{\mathrm{w}}\frac{\partial^2\bar{u}_{\mathrm{w}}}{\partial z^2} \quad \text{(A7)}$$

where C_{w}, $C_{\mathrm{v}r}^{\mathrm{w}}$ and $C_{\mathrm{vz}}^{\mathrm{w}}$ are shown in Eq. (6).

A2 Governing equation of air phase

The air flow of the unsaturated soil elemnt can be calculated by the mass flow rate of air in the radial and vertical directions $J_{\mathrm{a}r}$ and J_{az}, respectively. The net mass of the air flow of the elemnt is equal to the difference in mass between the air flowing into and out of the soil in a period of time:

$$\frac{\partial M_{\mathrm{a}}}{\partial t}=-\left(\frac{\partial J_{\mathrm{a}r}}{\partial r}+\frac{\partial J_{\mathrm{az}}}{\partial z}\right)r\mathrm{d}r\mathrm{d}\theta=-\left(\frac{\partial J_{\mathrm{a}r}}{\partial r}+\frac{\partial J_{\mathrm{az}}}{\partial z}\right)V_0 \quad \text{(A8)}$$

where M_{a} is the mass of the air in the soil, and $M_{\mathrm{a}}=V_{\mathrm{a}}\rho_{\mathrm{a}}$; V_{a} and ρ_{a} are the volume and density of the air, respectively.

Rearranging Eq. (A8) gives

$$\frac{\partial\left(M_{\mathrm{a}}/V_0\right)}{\partial t}=-\left(\frac{\partial J_{\mathrm{a}r}}{\partial r}+\frac{\partial J_{\mathrm{az}}}{\partial z}\right) \quad \text{(A9)}$$

Assuming that the air flow in unsaturated soil conforms to Fick's law:

$$J_{\mathrm{a}r}=-D_{\mathrm{a}r}^{\bullet}\frac{\partial u_{\mathrm{a}}}{\partial r}=-\frac{k_{\mathrm{a}r}}{g}\frac{\partial u_{\mathrm{a}}}{\partial r} \quad \text{(A10)}$$

$$J_{\mathrm{az}}=-D_{\mathrm{az}}^{\bullet}\frac{\partial u_{\mathrm{a}}}{\partial z}=-\frac{k_{\mathrm{az}}}{g}\frac{\partial u_{\mathrm{a}}}{\partial z} \quad \text{(A11)}$$

where $D_{\mathrm{a}r}^{\bullet}$ and $D_{\mathrm{az}}^{\bullet}$ are the corrected radial and vertical air flow conductivity constants in the soil.

Taking derivatives of Eqs. (A10) and (A11) with respect to *r* and *z*, respectively, and then substituting them into Eq. (A9) leads to

$$\frac{\partial\left(V_{\mathrm{a}}/V_0\right)}{\partial t}+\frac{V_{\mathrm{a}}}{\rho_{\mathrm{a}}V_0}\frac{\partial\rho_{\mathrm{a}}}{\partial t}= \frac{k_{\mathrm{a}r}}{g\rho_{\mathrm{a}}}\left(\frac{\partial^2u_{\mathrm{a}}}{\partial r^2}+\frac{1}{r}\frac{\partial u_{\mathrm{a}}}{\partial r}\right)+\frac{k_{\mathrm{az}}}{g\rho_{\mathrm{a}}}\frac{\partial^2u_{\mathrm{a}}}{\partial z^2} \quad \text{(A12)}$$

The air is defined as an ideal air, so there is

$$\rho_{\mathrm{a}}=\frac{\tilde{u}_{\mathrm{a}}M}{RT_{\mathrm{at}}} \quad \text{(A13)}$$

Substituting Eq. (A13) into Eq. (A12) yields

$$\frac{\partial(V_{\mathrm{a}}/V_0)}{\partial t}+\frac{V_{\mathrm{a}}}{\tilde{u}_{\mathrm{a}}V_0}\frac{\partial u_{\mathrm{a}}}{\partial t}=$$

$$\frac{RT}{M}\frac{k_{ar}}{g\tilde{u}_{\mathrm{a}}}\left(\frac{\partial^2 u_{\mathrm{a}}}{\partial r^2}+\frac{1}{r}\frac{\partial u_{\mathrm{a}}}{\partial r}\right)+\frac{RT}{M}\frac{k_{\mathrm{az}}}{g\tilde{u}_{\mathrm{a}}}\frac{\partial^2 u_{\mathrm{a}}}{\partial z^2} \tag{A14}$$

Equation (A14) is a non-linear equation, and some of the coefficients depend on the pressure of the air. However, when the excess pore-air pressure is much lower than the atmospheric pressure, the absolute pressure $\tilde{u}_{\mathrm{a}}$ can be approximated as a constant $\tilde{u}_{\mathrm{a}}^0$.

Based on Boyle's Law, we have

$$\frac{V_{\mathrm{a}}}{\overline{V}_{\mathrm{a}}^0}=\frac{\tilde{u}_{\mathrm{a}}^0}{\tilde{u}_{\mathrm{a}}} \tag{A15}$$

where $\overline{V}_{\mathrm{a}}^0$ is the initial air volume before consolidation.

From Eq. (A15), we can get

$$\frac{V_{\mathrm{a}}}{\tilde{u}_{\mathrm{a}}V_0}=\frac{u_{\mathrm{atm}}}{\left(\tilde{u}_{\mathrm{a}}^0\right)^2}\left(1-S_{\mathrm{r}0}\right)n_0 \tag{A16}$$

In the same way, by taking the derivative of Eq. (A6) with respect to *t*, and combining with Eqs. (A14) and (A16), the governing equation of the air phase can be obtained as follows:

$$\frac{\partial\overline{u}_{\mathrm{a}}}{\partial t}=-C_{\mathrm{a}}\frac{\partial\overline{u}_{\mathrm{w}}}{\partial t}-C_{\mathrm{v}r}^{\mathrm{a}}\left(\frac{\partial^2 u_{\mathrm{a}}}{\partial r^2}+\frac{1}{r}\frac{\partial u_{\mathrm{a}}}{\partial r}\right)-C_{\mathrm{vz}}^{\mathrm{a}}\frac{\partial^2\overline{u}_{\mathrm{a}}}{\partial z^2} \tag{A17}$$

Equations for calculating C_{a}, $C_{\mathrm{v}r}^{\mathrm{a}}$ and $C_{\mathrm{vz}}^{\mathrm{a}}$ are the same as Eqs. (3) to (5).

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