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Analytical solutions for 1D consolidation of unsaturated soils with mixed nonhomogeneous boundary conditions

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Abstract: The consolidation of unsaturated soil is of great significance to road engineering, soft foundation soil improvement, etc. Based on the one-dimensional consolidation theory of unsaturated soil proposed by Fredlund and Hasan, the governing equations for pore water pressure and pore air pressure in the soil are established. The initial conditions and a type of time-dependent mixed nonhomogeneous boundary conditions of single-layer unsaturated soil are presented which constitutes the solution of 1D consolidation of unsaturated soil. The homogenization of nonhomogeneous boundary conditions and the eigenfunction expansion method are adopted to derive exact analytical solutions in time domain for the dissipation of pore water pressure and pore air pressure in the soil. Finally, the method proposed in this paper is validated by comparison with published results, and several examples are used to analyze the effects of exponentially changing boundary conditions on the dissipation of pore water pressure, pore air pressure, and deformation of unsaturated soils. The results show that the speed of exponential change of pore pressure on the boundary or flux across the boundary has significant effect on the consolidation process of unsaturated soil.

Keywords: consolidation of unsaturated soil; mixed nonhomogeneous boundary conditions; analytical solutions; eigenfunction expansion method; exponentially changing boundary conditions

1 Introduction

Soil consolidation and settlement are very common in geotechnical projects, and have always been the focus of many engineers. Because of the wide distribution of unsaturated soil in practical engineering, Bishop et al.[1], Blight[2], Barden[3], Alonso et al.[4], Fredlund et al.[5–7], Yang[8], Chen[9–10] and Yin[11] have studied the deformation of unsaturated soil extensively. At present, the one-dimensional consolidation theory proposed by Fredlund et al.[5] which assumes the continuity of liquid phase and gas phase, and the three-dimensional consolidation theory developed by Dakshanamurthy et al.[6], which further extended Fredlund's theory, are among the most commonly used analytical theories for consolidation of unsaturated soil.

Based on the assumption that the permeability coefficient and volume variation coefficient of unsaturated soil in Fredlund's equations are constant, many researchers have derived the analytical solutions of consolidation of unsaturated soil under different homogeneous boundary conditions. Ho et al.[12–15] assumed that pore water and pore air are permeable on the top surface, or both surfaces of the single-layer unsaturated soil, and then the series solutions for one-dimensional or two-dimensional consolidation of unsaturated soil is developed by adopting the eigenfunction expansion method. Qin et al.[16–19] simplified the consolidation governing equation in the Laplace transform domain by introducing gas and liquid flow rates, and analytically solved the case of unsaturated soil with only water/air flow or only air flow exits on the top surface. Wang et al.[20–21] decoupled the governing equations of pore water pressure and pore gas pressure by introducing intermediate variables, and then combined Laplace transform with Crump's numerical inverse transformation method, they obtained the semi-analytical solutions for soil consolidation, and then analyzed the consolidation process of unsaturated soil with semi-permeable boundary. Zhou et al.[22] used the differential quadrature method to analyze the effects of complex initial pore water pressure and air pressure distributions and arbitrary homogeneous boundary conditions on the consolidation of unsaturated soil. Cheng et al.[23] achieved the analytical solution of the two-dimensional consolidation problem of unsaturated soil by using the introduced function and separation of variables. It can be concluded that there has been a lot of progress in the research of in the consolidation of unsaturated soil under homogeneous boundary conditions. However, the boundary of unsaturated soil layer is not necessarily completely permeable or impermeable for pore water and pore air in engineering projects, which means that

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the boundary condition of soil layer may change with time. It's a non-homogeneous boundary. Up to now, researchers have given the analytical solutions of consolidation of unsaturated soil under several kinds of non-homogeneous boundary conditions. Shan et al. [24–25] gave the series of solution of consolidation of unsaturated soil under four kinds of non-homogeneous boundary conditions by using eigenfunction method, and analyzed the influence of periodic boundary on the consolidation of unsaturated soil. But there are almost no cyclic boundary conditions of pore pressures in engineering practice. Zhou et al.[26], Wang et al.[27] and Huang et al.[28] all give the consolidation solutions of the dissipation of pore pressure in the soil when the pore water pressure and pore gas pressure on the upper and lower boundaries of the unsaturated soil increase exponentially over time.

A number of references[26–28] pointed out that the exponential boundary condition can better reflect the continuous change of the boundary of unsaturated soil from permeable and breathable conditions to impermeable and non-breathable conditions, but there is a deflect that it can't precisely express the change of pore pressures at the impermeable or non-breathable boundaries[26–28]. The variations of pore water pressure or pore air pressure at the impermeable or non-breathable surface are affected by the consolidation process, rather than directly specified by the exponential functions in references[26–28], which may lead to inconsistency. In this paper, one type of mixed non-homogeneous boundary conditions and arbitrary initial conditions are given premised on the one-dimensional consolidation theory of unsaturated soil proposed by Fredlund et al. [5]. The pore pressure or pore pressure gradient on the boundary can be expressed as an arbitrary exponential function with respect to time. Compared with the boundary conditions proposed in references[26–28], it can better reveal the real boundary conditions for pore water or pore air on the impermeable surface. Then, by using the analytical method proposed by Shan et al.[25], the exact time-domain analytical solutions of pore water pressure and pore air pressure are given by homogenizationn of nonhomogeneous boundary conditions and the method of eigenfunction expansion. Finally, the analytical method is verified by a numerical example, and the variations of pore water pressure and pore air pressure in unsaturated soil as well as the compressive settlement of soil are analyzed under different exponential boundary conditions.

2 Governing equation

The consolidation theory of unsaturated soil proposed by Fredlund et al.[5] is premised on the following assumptions: (1) the pore air and pore water are continuous; (2) the soil particles and pore water are incompressible; (3) the dissolution of pore air in water and the movement of water vapor are not considered; and (4) the related volume change coefficient and permeability coefficient of unsaturated soil remain constant during the consolidation process. In this paper, the derived one-dimensional consolidation governing equation [5] is given directly:

$$\left(C_{\mathrm{v}}^{\mathrm{w}}\frac{\partial^2}{\partial x^2}+\frac{\partial}{\partial t}\right)u_{\mathrm{w}}+C_{\mathrm{w}}\frac{\partial u_{\mathrm{a}}}{\partial t}=0 \tag{1}$$

$$C_{\mathrm{a}}\frac{\partial u_{\mathrm{w}}}{\partial t}+\left(C_{\mathrm{v}}^{\mathrm{a}}\frac{\partial^2}{\partial x^2}+\frac{\partial}{\partial t}\right)u_{\mathrm{a}}=0 \tag{2}$$

where u_{w} and u_{a} are pore water pressure and pore gas pressure, respectively.

$$C_{\mathrm{w}}=\frac{m_1^{\mathrm{w}}-m_2^{\mathrm{w}}}{m_2^{\mathrm{w}}},\ C_{\mathrm{a}}=\frac{\bar{u}_{\mathrm{a}}m_2^{\mathrm{a}}}{\bar{u}_{\mathrm{a}}m_1^{\mathrm{a}}-\bar{u}_{\mathrm{a}}m_2^{\mathrm{a}}-n(1-S)}$$
$$C_{\mathrm{v}}^{\mathrm{w}}=\frac{k_{\mathrm{w}}}{\gamma_{\mathrm{w}}m_2^{\mathrm{w}}},\ C_{\mathrm{v}}^{\mathrm{a}}=\frac{k_{\mathrm{a}}RT}{\omega_{\mathrm{a}}g\left[\bar{u}_{\mathrm{a}}m_1^{\mathrm{a}}-\bar{u}_{\mathrm{a}}m_2^{\mathrm{a}}-n(1-S)\right]} \tag{3}$$

where k_{w} and k_{a} are the permeability coefficients of pore water and pore air, respectively; m_1^{w} and m_1^{a} are the volume changes of pore water and pore air caused by the change of unit vertical net stress ($\sigma-u_{\mathrm{a}}$) at K_0 condition, respectively; m_2^{w} and m_2^{a} are the volume changes of pore water and pore air caused by the change of unit matric suction ($u_{\mathrm{a}}-u_{\mathrm{w}}$), respectively; γ_{w} is the unit weight of water; S is the degree of saturation; n is the porosity; ω_a is the mass of air molecules; R is the general air constant; T is the absolute temperature; $\bar{u}_{\mathrm{a}}$ is the absolute pore air pressure, $\bar{u}_{\mathrm{a}}=u_{\mathrm{a}}^0+\bar{u}_{\mathrm{atm}}$, u_{a}^0 is the initial pore air pressure, $\bar{u}_{\mathrm{atm}}$ is the atmospheric pressure; and g is the acceleration of gravity.

Equations (1) and (2) are expressed in matrix form as

$$\boldsymbol{Mu}_{,xx}+\boldsymbol{Cu}_{,t}=0 \tag{4}$$

where $(\ )_{,x}$ $(\ )_{,t}$ represents the derivative of depth x and time t, respectively, and

$$\boldsymbol{u}=\begin{Bmatrix}u_{\mathrm{w}}\\u_a\end{Bmatrix},\ \boldsymbol{M}=\begin{bmatrix}C_{\mathrm{v}}^{\mathrm{w}}&0\\0&\dfrac{C_{\mathrm{v}}^{\mathrm{a}}C_{\mathrm{w}}}{C_{\mathrm{a}}}\end{bmatrix},\ \boldsymbol{C}=\begin{bmatrix}1&C_{\mathrm{w}}\\C_{\mathrm{w}}&\dfrac{C_{\mathrm{w}}}{C_{\mathrm{a}}}\end{bmatrix} \tag{5}$$

3 Initial conditions and boundary conditions

The model for one-dimensional consolidation of unsaturated soil used in this paper is shown in Fig.1. Any point on the surface of unsaturated soil layer is selected as the origin, and the direction along the depth is x-axis, the coordinate system is established, and all deformation and seepages occur along the x-axis direction. H is the thickness of unsaturated soil layer.

The following arbitrary initial conditions and non-homogeneous boundary conditions are adopted for the unsaturated soil layer:

$$u_{\mathrm{w}}\left(x,0\right)=g_1(x),\ \ u_{\mathrm{a}}\left(x,0\right)=g_2(x) \tag{6}$$

$$\left.\begin{aligned} &u_{\mathrm{w}}\left(0,t\right)=f_1(t),\ \ u_{\mathrm{a}}\left(0,t\right)=f_2(t) \\ &u_{\mathrm{w}}\left(H,t\right)=f_3(t),\ u_{\mathrm{a},x}\left(H,t\right)=f_4(t) \end{aligned}\right\} \tag{7}$$

where $g_1(x)$ and $g_2(x)$ represent the distribution of pore water pressure and pore gas pressure along the depth direction at the initial moment, respectively. $f_i(t)\left(i=1,2,3,4\right)$ is an arbitrarily specified function with regard to time, which represents the variation of pore water pressure and pore gas pressure on the surface of the unsaturated soil, pore water pressure and gradient of pore gas pressure on the bottom of the unsaturated soil over time, respectively. When $f_i(t)=0$, the non-homogeneous boundary conditions become homogeneous, indicating that the surface of the unsaturated soil is permeable for pore water and pore air, but the bottom is permeable for pore water and impermeable for pore air.

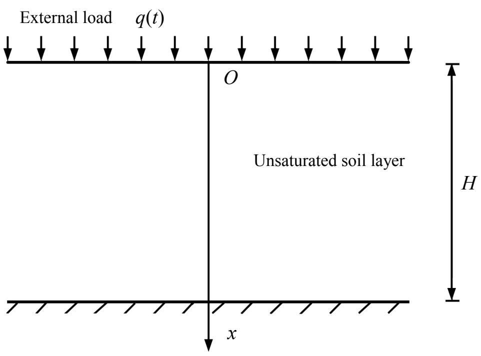

Fig. 1 Model for one-dimensional consolidation of single-layer unsaturated soil

4 Derivation of the solutions

In order to obtain the solution of one-dimensional consolidation of unsaturated soil under non-homogeneous boundary conditions, $\boldsymbol{u}$ is decomposed into the following two parts:

$$\boldsymbol{u}=\boldsymbol{u}^{\mathrm{d}}+\boldsymbol{u}^{\mathrm{s}} \tag{8}$$

wher $\boldsymbol{u}^{\mathrm{s}}$ is a function specified to satisfy the non-homogeneous boundary condition (7) and can be assumed as the following form:

$$\boldsymbol{u}^{\mathrm{s}}=\begin{Bmatrix} \left[f_3(t)-f_1(t)\right]x/H+f_1(t) \\ f_4(t)x+f_2(t) \end{Bmatrix} \tag{9}$$

By substituting the Eq. (8) into Eqs. (4), (6), and (7), the governing equation expressed in $\boldsymbol{u}^{\mathrm{d}}$ can be obtained as

$$\boldsymbol{M}\boldsymbol{u}^{\mathrm{d}}_{,xx}+\boldsymbol{C}\boldsymbol{u}^{\mathrm{d}}_{,t}=-\boldsymbol{M}\boldsymbol{u}^{\mathrm{d}}_{,xx}-\boldsymbol{C}\boldsymbol{u}^{\mathrm{s}}_{,t} \tag{10}$$

and the initial and boundary conditions as follows:

$$\boldsymbol{u}^{\mathrm{d}}\left(x,0\right)=\boldsymbol{u}\left(x,0\right)-\boldsymbol{u}^{\mathrm{s}}\left(x,0\right) \tag{11}$$

$$\left.\begin{aligned} &u^{\mathrm{d}}_{\mathrm{w}}\left(0,t\right)=0,\ u^{\mathrm{d}}_{\mathrm{a}}\left(0,t\right)=0 \\ &u^{\mathrm{d}}_{\mathrm{w}}\left(H,t\right)=0,\ u^{\mathrm{d}}_{\mathrm{a},x}\left(H,t\right)=0 \end{aligned}\right\} \tag{12}$$

Therefore, the problem of solving the homogeneous differential equations on $\boldsymbol{u}$ with non-homogeneous boundary conditions can be transformed into the problem of solving non-homogeneous differential equation on $\boldsymbol{u}^{\mathrm{d}}$ with homogeneous boundary conditions.

4.1 Eigenfunctions and eigenvalues of the equation

In order to solve the non-homogeneous Eq. (10) by using the eigenfunction expansion method, we first derive the eigenfunctions from the homogeneous form of Eq.(10).

$$\boldsymbol{M}\boldsymbol{u}^{\mathrm{d}}_{,xx}+\boldsymbol{C}\boldsymbol{u}^{\mathrm{d}}_{,t}=0 \tag{13}$$

By separating variables, the solution of Eq. (13) can be assumed to be

$$\boldsymbol{u}^{\mathrm{d}}=\boldsymbol{Z}^{\mathrm{d}}\left(x\right)\mathrm{e}^{-\omega^2 t} \tag{14}$$

where $\boldsymbol{Z}^{\mathrm{d}}(x)=\left\{z^{\mathrm{d}}_{\mathrm{w}}(x),z^{\mathrm{d}}_{\mathrm{a}}(x)\right\}^{\mathrm{T}}$, ω is a non-negative real number.

Substituting Eq. (14) into the boundary condition (12), we can obtain

$$z^{\mathrm{d}}_{\mathrm{w}}(0)=0,\ z^{\mathrm{d}}_{\mathrm{a}}(0)=0,\ z^{\mathrm{d}}_{\mathrm{w}}(H)=0,\ z^{\mathrm{d}}_{\mathrm{a},x}(H)=0 \tag{15}$$

By substituting Eq. (14) into Eq. (13), we can obtain

$$\boldsymbol{M}\boldsymbol{Z}^{\mathrm{d}}_{,xx}\left(x\right)-\omega^2\boldsymbol{C}\boldsymbol{Z}^{\mathrm{d}}\left(x\right)=0 \tag{16}$$

When ω=0, Eq. (16) becomes

$$\boldsymbol{M}\boldsymbol{Z}^{\mathrm{d}}_{,xx}\left(x\right)=0 \tag{17}$$

Since the determinant of matrix $\boldsymbol{M}$ is not 0, the solution of Eq. (17) is

$$\boldsymbol{Z}^{\mathrm{d}}_0\left(x\right)=\begin{Bmatrix} c_{01} \\ c_{02} \end{Bmatrix}x+\begin{Bmatrix} d_{01} \\ d_{02} \end{Bmatrix} \tag{18}$$

By substituting Eq. (18) into the boundary condition (15), the expressions of the unknown constants c_{01}, c_{02}, d_{01} and d_{02} can be obtained to be 0.Thus

$$\boldsymbol{Z}^{\mathrm{d}}_0\left(x\right)=0 \tag{19}$$

When ω>0, the solution of Eq. (16) can be expressed as

$$\boldsymbol{Z}^{\mathrm{d}}\left(x\right)=\boldsymbol{F}\mathrm{e}^{\beta x} \tag{20}$$

where β and $\boldsymbol{F}$ are unknown constant and unknown second-order column vector, respectively. Substituting Eq.(20) into Eq.(16) yields

$$\left(\beta^2\boldsymbol{M}-\omega^2\boldsymbol{C}\right)\boldsymbol{F}=0 \tag{21}$$

Its specific form is

$$\begin{bmatrix} C^{\mathrm{w}}_{\mathrm{v}}\beta^2-\omega^2 & -C_{\mathrm{w}}\omega^2 \\ -C_{\mathrm{w}}\omega^2 & \left(C^{\mathrm{w}}_{\mathrm{v}}\beta^2-\omega^2\right)C_{\mathrm{w}}/C_{\mathrm{a}} \end{bmatrix}\boldsymbol{F}=0 \tag{22}$$

If and only if the determinant of the coefficient matrix of Eq. (22) is 0, $\boldsymbol{F}$ will have a non-zero solution, then, $\boldsymbol{Z}^{\mathrm{d}}(x)$ will have a non-zero solution, from which we can obtain the determinant

$$y^4 - by^2 + c = 0 \tag{23}$$

where

$$\left.\begin{aligned} &y = \beta / \omega,\ b = \left(C_{\mathrm{v}}^{\mathrm{a}} + C_{\mathrm{v}}^{\mathrm{w}}\right) / \left(C_{\mathrm{v}}^{\mathrm{a}} C_{\mathrm{v}}^{\mathrm{w}}\right) \\ &c = \left(1 - C_{\mathrm{a}} C_{\mathrm{w}}\right) / \left(C_{\mathrm{v}}^{\mathrm{a}} C_{\mathrm{v}}^{\mathrm{w}}\right) \end{aligned}\right\} \tag{24}$$

From Eq. (23), we can get two roots of y^2, denoted as A and B:

$$A = \left(b + \sqrt{\Delta}\right)/2,\ B = \left(b - \sqrt{\Delta}\right)/2,\ \Delta = b^2 - 4c \tag{25}$$

Substituting the parameters of unsaturated soil given by Fredlund and other scholars [7, 12–25] into Eq. (24), we can find that they all satisfy $c > 0$ and $b < 0$. Then substituting them into Eq. (25), we can get $\Delta > 0$, $A < 0$, $B < 0$. In this paper, the solution is given for this case, so the four roots of y are

$$\left.\begin{aligned} &y_1 = \mathrm{i}\alpha_1 = \mathrm{i}\sqrt{-A},\ y_3 = -\mathrm{i}\alpha_1 \\ &y_2 = \mathrm{i}\alpha_2 = \mathrm{i}\sqrt{-B},\ y_4 = -\mathrm{i}\alpha_2 \end{aligned}\right\} \tag{26}$$

Rewrite the Eq. (22) into the following form

$$\begin{bmatrix} C_{\mathrm{v}}^{\mathrm{w}} y_j^2 - 1 & -C_{\mathrm{w}} \\ -C_{\mathrm{w}} & \left(C_{\mathrm{v}}^{\mathrm{w}} y_j^2 - 1\right) C_{\mathrm{w}} / C_{\mathrm{a}} \end{bmatrix} \boldsymbol{F}_j = 0 \tag{27}$$

Substituting four roots y_j of y in Eq. (26) into Eq. (27), the corresponding eigenvector $\boldsymbol{F}_j = \left\{F_{j1}, F_{j2}\right\}^{\mathrm{T}}$ can be obtained, respectively:

$$\boldsymbol{F}_1 = A_1 \boldsymbol{G}_1, \boldsymbol{F}_2 = A_2 \boldsymbol{G}_2, \boldsymbol{F}_3 = A_3 \boldsymbol{G}_1, \boldsymbol{F}_4 = A_4 \boldsymbol{G}_2 \tag{28}$$

$$\left.\begin{aligned} &\boldsymbol{G}_1 = \begin{Bmatrix} G_{11} \\ G_{12} \end{Bmatrix} = \begin{Bmatrix} C_{\mathrm{w}} \\ C_{\mathrm{v}}^{\mathrm{w}} A - 1 \end{Bmatrix} \\ &\boldsymbol{G}_2 = \begin{Bmatrix} G_{21} \\ G_{22} \end{Bmatrix} = \begin{Bmatrix} C_{\mathrm{w}} \\ C_{\mathrm{v}}^{\mathrm{w}} B - 1 \end{Bmatrix} \end{aligned}\right\} \tag{29}$$

where A_1, A_2, A_3, A_4 are undetermined constants. Therefore, according to Eq. (20), we have the solution to Eq. (16):

$$\begin{aligned} \boldsymbol{Z}^{\mathrm{d}}(x) &= \boldsymbol{F}_1 \mathrm{e}^{\mathrm{i}\alpha_1 \omega x} + \boldsymbol{F}_2 \mathrm{e}^{\mathrm{i}\alpha_2 \omega x} + \boldsymbol{F}_3 \mathrm{e}^{-\mathrm{i}\alpha_1 \omega x} + \boldsymbol{F}_4 \mathrm{e}^{-\mathrm{i}\alpha_2 \omega x} = \\ &\boldsymbol{G}_1 \left(A_1 \mathrm{e}^{\mathrm{i}\alpha_1 \omega x} + A_3 \mathrm{e}^{-\mathrm{i}\alpha_1 \omega x}\right) + \boldsymbol{G}_2 \left(A_2 \mathrm{e}^{\mathrm{i}\alpha_2 \omega x} + A_4 \mathrm{e}^{-\mathrm{i}\alpha_2 \omega x}\right) \end{aligned} \tag{30}$$

To make $\boldsymbol{Z}^{\mathrm{d}}(x)$ a real function, A_i needs to take conjugate complex numbers and take

$$A_1 = \frac{b_1 - \mathrm{i}b_2}{2},\ A_2 = \frac{b_3 - \mathrm{i}b_4}{2},\ A_3 = \overline{A_1},\ A_4 = \overline{A_2} \tag{31}$$

where b_i (i = 1,2,3,4) is the real integral constant to be determined. Substitution of Eq. (31) into Eq. (30) results in

$$\begin{aligned} \boldsymbol{Z}^{\mathrm{d}}(x) = \boldsymbol{G}_1 \left[b_1 \cos\left(\omega\alpha_1 x\right) + b_2 \sin\left(\omega\alpha_1 x\right)\right] + \\ \boldsymbol{G}_2 \left[b_3 \cos\left(\omega\alpha_2 x\right) + b_4 \sin\left(\omega\alpha_2 x\right)\right] \end{aligned} \tag{32}$$

When $\omega > 0$, substituting Eq. (32) into boundary condition Eq. (15) gives

$$\begin{bmatrix} G_{11} & 0 & G_{21} & 0 \\ G_{12} & 0 & G_{22} & 0 \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{33}$$

where

$$\left.\begin{aligned} &a_{31} = G_{11} \cos\left(\omega\alpha_1 H\right), a_{32} = G_{11} \sin\left(\omega\alpha_1 H\right) \\ &a_{33} = G_{21} \cos\left(\omega\alpha_2 H\right), a_{34} = G_{21} \sin\left(\omega\alpha_2 H\right) \\ &a_{41} = -\alpha_1 G_{12} \sin\left(\omega\alpha_1 H\right), a_{42} = \alpha_1 G_{12} \cos\left(\omega\alpha_1 H\right) \\ &a_{43} = -\alpha_2 G_{22} \sin\left(\omega\alpha_2 H\right), a_{44} = \alpha_2 G_{22} \cos\left(\omega\alpha_2 H\right) \end{aligned}\right\} \tag{34}$$

If and only if the determinant of the coefficient matrix of Eq. (33) is 0, b_i (i =1,2,3,4) will have a non-zero solution, thus, after calculating the matrix's determinant, the equation satisfied by the eigenvalue ω is obtained:

$$\frac{\alpha_2 G_{11} G_{22}}{\alpha_1 G_{12} G_{21}} \tan\left(\omega\alpha_1 H\right) \cot\left(\omega\alpha_2 H\right) = 1 \tag{35}$$

Eq. (35) is a transcendental equation, so ω has infinite roots. From small to large, it is denoted as ω_k (k=1,2…) successively. Each root ω_k corresponds to a set of solutions of $b_{1\mathrm{k}}$, $b_{2\mathrm{k}}$, $b_{3\mathrm{k}}$ and $b_{4\mathrm{k}}$. They can be obtained from Eq. (33), as set below:

$$\left.\begin{aligned} &b_{1k} = 0,\ \ b_{2k} = G_{21} \sin\left(\alpha_2 \omega_k H\right) \\ &b_{3k} = 0,\ \ b_{4k} = -G_{11} \sin\left(\alpha_1 \omega_k H\right) \end{aligned}\right\} \tag{36}$$

Considering Eqs. (19), (32), (35) and (36), the eigenfunction corresponding to the k^{th} eigenvalue ω_k of the homogeneous Eq.(13) is expressed as

$$\begin{aligned} \boldsymbol{Z}_k^{\mathrm{d}}(x) = \boldsymbol{G}_1 b_{2k} \sin\left(\omega_k \alpha_1 x\right) + \boldsymbol{G}_2 b_{4k} \sin\left(\omega_k \alpha_2 x\right) \\ k = 1, 2, \cdots \end{aligned} \tag{37}$$

where $\boldsymbol{Z}_k^{\mathrm{d}}(x) = \left\{z_{\mathrm{w}k}^{\mathrm{d}}(x), z_{\mathrm{a}k}^{\mathrm{d}}(x)\right\}^{\mathrm{T}}$.

4.2 Orthogonality of eigenfunctions

Let ω_p and ω_q be the two different eigenvalues of Eq. (16), and the corresponding eigenfunctions are denoted as $\boldsymbol{Z}_p^{\mathrm{d}}(x)$ and $\boldsymbol{Z}_q^{\mathrm{d}}(x)$, respectively. According to Eq.(16), which ω_p, ω_q, $\boldsymbol{Z}_p^{\mathrm{d}}(x)$ and $\boldsymbol{Z}_q^{\mathrm{d}}(x)$ satisfy, we multiply the term $\left[\boldsymbol{Z}_q^{\mathrm{d}}(x)\right]^{\mathrm{T}}$ and $\left[\boldsymbol{Z}_p^{\mathrm{d}}(x)\right]^{\mathrm{T}}$ on the left side of the two equations, respectively, and then we can get

$$\left[\boldsymbol{Z}_q^{\mathrm{d}}\right]^{\mathrm{T}} \boldsymbol{M} \boldsymbol{Z}_{p,xx}^{\mathrm{d}}(x) - \omega_p^2 \left[\boldsymbol{Z}_q^{\mathrm{d}}\right]^{\mathrm{T}} \boldsymbol{C} \boldsymbol{Z}_p^{\mathrm{d}}(x) = 0 \tag{38}$$

$$\left[\boldsymbol{Z}_p^{\mathrm{d}}\right]^{\mathrm{T}} \boldsymbol{M} \boldsymbol{Z}_{q,xx}^{\mathrm{d}}(x) - \omega_q^2 \left[\boldsymbol{Z}_p^{\mathrm{d}}\right]^{\mathrm{T}} \boldsymbol{C} \boldsymbol{Z}_q^{\mathrm{d}}(x) = 0 \tag{39}$$

From the symmetry of matrix $\boldsymbol{C}$, we can know that:

$$\left[\boldsymbol{Z}_q^{\mathrm{d}}\right]^{\mathrm{T}}\boldsymbol{C}\boldsymbol{Z}_p^{\mathrm{d}}(x)=\left[\boldsymbol{Z}_p^{\mathrm{d}}\right]^{\mathrm{T}}\boldsymbol{C}\boldsymbol{Z}_q^{\mathrm{d}}(x) \tag{40}$$

As a result, after subtracting Eq. (38) from Eq. (39) and integrating the subtracted equation from 0 to H with respect to x, we can get

$$\int_0^H\left\{\left[\boldsymbol{Z}_q^{\mathrm{d}}\right]^{\mathrm{T}}\boldsymbol{M}\boldsymbol{Z}_{p,xx}^{\mathrm{d}}-\left[\boldsymbol{Z}_p^{\mathrm{d}}\right]^{\mathrm{T}}\boldsymbol{M}\boldsymbol{Z}_{q,xx}^{\mathrm{d}}\right\}\mathrm{d}x=\left(\omega_p^2-\omega_q^2\right)\int_0^H\left[\boldsymbol{Z}_p^{\mathrm{d}}\right]^{\mathrm{T}}\boldsymbol{C}\boldsymbol{Z}_q^{\mathrm{d}}\mathrm{d}x \tag{41}$$

We can simplify the integral terms on the left side of Eq. (41) by partial integration method. Due to the symmetry of matrix $\boldsymbol{M}$, we can deduce that

$$\begin{aligned}&\int_0^H\left\{\left[\boldsymbol{Z}_q^{\mathrm{d}}\right]^{\mathrm{T}}\boldsymbol{M}\boldsymbol{Z}_{p,xx}^{\mathrm{d}}-\left[\boldsymbol{Z}_p^{\mathrm{d}}\right]^{\mathrm{T}}\boldsymbol{M}\boldsymbol{Z}_{q,xx}^{\mathrm{d}}\right\}\mathrm{d}x=\\&\left[\left[\boldsymbol{Z}_q^{\mathrm{d}}\right]^{\mathrm{T}}\boldsymbol{M}\boldsymbol{Z}_{p,x}^{\mathrm{d}}-\left[\boldsymbol{Z}_p^{\mathrm{d}}\right]^{\mathrm{T}}\boldsymbol{M}\boldsymbol{Z}_{q,x}^{\mathrm{d}}\right]_0^H=\\&C_{\mathrm{v}}^{\mathrm{w}}\begin{bmatrix}z_{\mathrm{w}q}^{\mathrm{d}}(H)z_{\mathrm{w}p,x}^{\mathrm{d}}(H)-z_{\mathrm{w}p}^{\mathrm{d}}(H)z_{\mathrm{w}q,x}^{\mathrm{d}}(H)\\-z_{\mathrm{w}q}^{\mathrm{d}}(0)z_{\mathrm{w}p,x}^{\mathrm{d}}(0)+z_{\mathrm{w}p}^{\mathrm{d}}(0)z_{\mathrm{w}q,x}^{\mathrm{d}}(0)\end{bmatrix}+\\&\frac{C_{\mathrm{v}}^{\mathrm{a}}C_{\mathrm{w}}}{C_{\mathrm{a}}}\begin{bmatrix}z_{\mathrm{a}q}^{\mathrm{d}}(H)z_{\mathrm{a}p,x}^{\mathrm{d}}(H)-z_{\mathrm{a}p}^{\mathrm{d}}(H)z_{\mathrm{a}q,x}^{\mathrm{d}}(H)\\-z_{\mathrm{a}q}^{\mathrm{d}}(0)z_{\mathrm{a}p,x}^{\mathrm{d}}(0)+z_{\mathrm{a}p}^{\mathrm{d}}(0)z_{\mathrm{a}q,x}^{\mathrm{d}}(0)\end{bmatrix}\end{aligned} \tag{42}$$

By substituting the boundary conditions Eq. (15) into Eq. (42), it is clear that the term on the right side of Eq. (41) is equal to 0. Therefore, we can infer from Eq. (41) that:

$$\left(\omega_p^2-\omega_q^2\right)\int_0^H\left[\boldsymbol{Z}_p^{\mathrm{d}}\right]^{\mathrm{T}}\boldsymbol{C}\boldsymbol{Z}_q^{\mathrm{d}}\mathrm{d}x=0 \tag{43}$$

Furthermore, the orthogonality of eigenfunctions can be obtained from Eq. (43):

$$\int_0^H\left[\boldsymbol{Z}_p^{\mathrm{d}}(x)\right]^{\mathrm{T}}\boldsymbol{C}\boldsymbol{Z}_q^{\mathrm{d}}(x)\mathrm{d}x=\begin{cases}0,\ p\neq q\\G_p,\ p=q\end{cases} \tag{44}$$

where G_p is a constant. Eq. (44) represents that $\left[\boldsymbol{Z}_p^{\mathrm{d}}(x)\right]^{\mathrm{T}}$ and $\boldsymbol{Z}_q^{\mathrm{d}}(x)$ are orthogonal with respect to the matrix $\boldsymbol{C}$. When $p=q=1,2,\cdots$, the G_p can be obtained by

$$G_p=\int_0^H\left[\left(z_{\mathrm{w}p}^{\mathrm{d}}\right)^2+2C_{\mathrm{w}}z_{\mathrm{w}p}^{\mathrm{d}}z_{\mathrm{a}p}^{\mathrm{d}}+C_{\mathrm{w}}\left(z_{\mathrm{a}p}^{\mathrm{d}}\right)^2/C_{\mathrm{a}}\right]\mathrm{d}x \tag{45}$$

4.3 Solution of the non-homogeneous equation

According to the eigenfunction expansion method and the eigenfunctions obtained in Eq.(37), the non-homogeneous differential equation (10) on $\boldsymbol{u}^{\mathrm{d}}$ with initial conditions (11) and boundary conditions (12) can be solved. The solution of the differential equation (10) on $\boldsymbol{u}^{\mathrm{d}}$ has the following form:

$$\boldsymbol{u}^{\mathrm{d}}(x,t)=\sum_{k=1}^{\infty}\boldsymbol{Z}_k^{\mathrm{d}}(x)T_k(t) \tag{46}$$

where $T_k(t)$ is an unknown function.

Substituting Eq. (46) into Eq. (10) gives

$$\boldsymbol{M}\sum_{k=1}^{\infty}\boldsymbol{Z}_{k,xx}^{\mathrm{d}}(x)T_k(t)+\boldsymbol{C}\sum_{k=1}^{\infty}\boldsymbol{Z}_k^{\mathrm{d}}(x)T_{k,t}(t)=-\boldsymbol{M}\boldsymbol{u}_{,xx}^{\mathrm{s}}-\boldsymbol{C}\boldsymbol{u}_{,t}^{\mathrm{s}} \tag{47}$$

The Eq. (47) can be firstly simplified by using the Eq. (16), which eigenfunctions should satisfy. Then multiply the simplified equation by $\left[\boldsymbol{Z}_p^{\mathrm{d}}(x)\right]^{\mathrm{T}}$ on both sides of the equation. Finally integrating the derived equation with respect to x from 0 to H, we can obtain:

$$\sum_{k=1}^{\infty}\left[\omega_k^2T_k(t)+T_{k,t}(t)\right]\int_0^H\left[\boldsymbol{Z}_p^{\mathrm{d}}(x)\right]^{\mathrm{T}}\boldsymbol{C}\boldsymbol{Z}_k^{\mathrm{d}}(x)=\int_0^H\left[\boldsymbol{Z}_p^{\mathrm{d}}(x)\right]^{\mathrm{T}}(-\boldsymbol{M}\boldsymbol{u}_{,xx}^{\mathrm{s}}-\boldsymbol{C}\boldsymbol{u}_{,t}^{\mathrm{s}})\mathrm{d}x \tag{48}$$

By using the orthogonality of the eigenfunctions (44), Eq. (48) can be simplified into:

$$T_{k,t}(t)+\omega_k^2T_k(t)=S_k(t) \tag{49}$$

$$S_k(t)=\frac{1}{G_k}\int_0^H\left[\boldsymbol{Z}_k^{\mathrm{d}}(x)\right]^{\mathrm{T}}\left(-\boldsymbol{M}\boldsymbol{u}_{,xx}^{\mathrm{s}}-\boldsymbol{C}\boldsymbol{u}_{,t}^{\mathrm{s}}\right)\mathrm{d}x \tag{50}$$

By substituting the initial condition (11) into Eq. (46), we have

$$\sum_{k=1}^{\infty}\boldsymbol{Z}_k^{\mathrm{d}}(x)T_k(0)=\boldsymbol{u}(x,0)-\boldsymbol{u}^{\mathrm{s}}(x,0) \tag{51}$$

By multiplying $\left[\boldsymbol{Z}_k^{\mathrm{d}}\right]^{\mathrm{T}}\boldsymbol{C}$ on the two sides of Eq. (51), then integrating the equation with respect to x from 0 to H, and finally using the orthogonality of eigenfunctions (44), we can obtain:

$$T_k(0)=\frac{1}{G_k}\int_0^H\left[\boldsymbol{Z}_k^{\mathrm{d}}(x)\right]^{\mathrm{T}}\boldsymbol{C}\left[\boldsymbol{u}(x,0)-\boldsymbol{u}^{\mathrm{s}}(x,0)\right]\mathrm{d}x \tag{52}$$

From Eq. (52), it is clear that initial distributions of pore pressures affect the expression of $T_k(0)$. Thus, the solution to the differential equation (49) and the initial condition (52) is

$$T_k(t)=\mathrm{e}^{-\omega_k^2t}T_k(0)+\mathrm{e}^{-\omega_k^2t}\int_0^t\mathrm{e}^{\omega_k^2\xi}S_k(\xi)\mathrm{d}\xi \tag{53}$$

Finally, the series solution of $\boldsymbol{u}^{\mathrm{d}}$ is obtained by substituting Eq. (37) and (53) into Eq. (46). And then adding Eq. (46) to Eq. (9) yields the solution for pore pressures during one-dimensional consolidation of unsaturated soil with non-homogeneous boundary condition (7) and arbitrary initial condition (6):

$$\boldsymbol{u}=\begin{Bmatrix}\left[f_3(t)-f_1(t)\right]x/H+f_1(t)\\f_4(t)x+f_2(t)\end{Bmatrix}+\sum_{k=1}^{\infty}\boldsymbol{Z}_k^{\mathrm{d}}(x)T_k(t) \tag{54}$$

where the spatial eigenfunction $\boldsymbol{Z}_k^{\mathrm{d}}(x)$ is given by Eq.(37), and the time function $T_k(t)$ is given by Eq. (53).

5 Consolidation deformation of unsaturated soil layer

Fredlund et al.[7] pointed out that the constitutive relation of one-dimensional consolidation of soil skeleton under K_0 load is

$$\frac{\partial \varepsilon_v}{\partial t}=m_1^s\frac{\partial(\sigma-u_a)}{\partial t}+m_2^s\frac{\partial(u_a-u_w)}{\partial t} \tag{55}$$

where ε_v is the volumetric strain; σ is the total stress; m_1^s is the change of the volume of the soil skeleton which caused by the change of the net unit vertical stress $(\sigma-u_a)$ in K_0 condition; m_2^s is the change of the volume of the soil skeleton caused by the change of the unit vertical matrix suction (u_a-u_w); and m_1^s and m_2^s are derived from the following relations

$$m_1^s=m_1^a+m_1^w,\ m_2^s=m_2^a+m_2^w \tag{56}$$

Integrate Eq. (55) with respect to time from 0 to t, and then for x from 0 to H, the expression of $S(t)$ for surface settlement of unsaturated soil can be obtained as follows:

$$\begin{aligned}S(t)=\ &m_1^s\int_0^H\{[\sigma(x,t)-u_a(x,t)]-\\&[\sigma(x,0)-u_a(x,0)]\}\mathrm{d}x+\\&m_2^s\int_0^H\{[u_a(x,t)-u_w(x,t)]-[u_a(x,0)-u_w(x,0)]\}\mathrm{d}x\end{aligned} \tag{57}$$

Equation (57) reflects the relationship between vertical net stress, matrix suction and soil settlement on the surface. If a constant step load is exerted on the top surface of unsaturated soil, the total stress σ in all depths of the unsaturated soil remains unchanged during consolidation process, and the initial pore water pressure and pore air pressure caused by external load are set as u_w^0 and u_a^0, respectively. In this case, Eq. (57) can be simplified into

$$\begin{aligned}S(t)=\ &m_1^s u_a^0 H-m_2^s(u_a^0-u_w^0)H+\\&(m_2^s-m_1^s)\int_0^H u_a(x,t)\mathrm{d}x-m_2^s\int_0^H u_w(x,t)\mathrm{d}x\end{aligned} \tag{58}$$

The settlement of the soil layer can be achieved by substituting the solution (54) of pore water pressure and pore gas pressure into Eq.(58). If the pore water pressure and pore gas pressure dissipate to 0 after consolidation is completed, the final settlement of the soil is $S_0=m_1^s u_a^0 H-m_2^s(u_a^0-u_w^0)H$.

6 Case study

In this section, the analytical solution derived above is used to analyze the consolidation process of unsaturated soil under different exponential boundary conditions. In Eq. (54), the summation term of the series solutions takes the first 10 000 terms into account. If not specified, most of the examples in this paper adopt the same physical parameters as those listed in Qin et al.[17], that is

$$\begin{aligned}&n=0.50,\ S=0.80,\ H=10\ \text{m},\ \gamma_w=9.8\ \text{kN/m}^3,\\&m_1^w=-0.5\times10^{-4}\ \text{kPa}^{-1},\ m_2^w=-2.0\times10^{-4}\ \text{kPa}^{-1},\\&m_1^a=-2.0\times10^{-4}\ \text{kPa}^{-1},\ m_2^a=1.0\times10^{-4}\ \text{kPa}^{-1},\\&R=8.314\ 32\ \text{J/(mol}\cdot\text{K)},\ T=293.16\ \text{K},\\&\omega_a=29\times10^{-3}\ \text{kg/mol},\ k_a=k_w=10^{-8}\ \text{m/s}\end{aligned} \tag{59}$$

Qin et al. [17] pointed out in their paper that the pore water pressure and pore gas pressure caused by instantaneous load 100 kPa in unsaturated soil are the same along the depth direction, which are 40 kPa and 20 kPa, respectively. This paper adopts the conclusion of Qin et al.[17]. That is, the initial conditions are assumed:

$$u_w(x,0)=u_w^0=40\ \text{kPa},\ u_a(x,0)=u_a^0=20\ \text{kPa} \tag{60}$$

6.1 Verification

In order to verify the proposed analytical method in this paper, the initial conditions such as Eq. (60) and the homogeneous boundary conditions are adopted:

$$\left.\begin{aligned}&u_w(0,t)=0,\ u_a(0,t)=0\\&u_w(H,t)=0,\ u_{a,x}(H,t)=0\end{aligned}\right\} \tag{61}$$

The pore water pressure and pore gas pressure in soil are calculated and compared by using the analytical solution method in this paper and the semi-analytical solution method proposed by Qin et al. [17].

The comparisons between pore water pressure and pore gas pressure obtained by two analytical methods, respectively are illustrated in Figs. 2 and 3. Obviously,

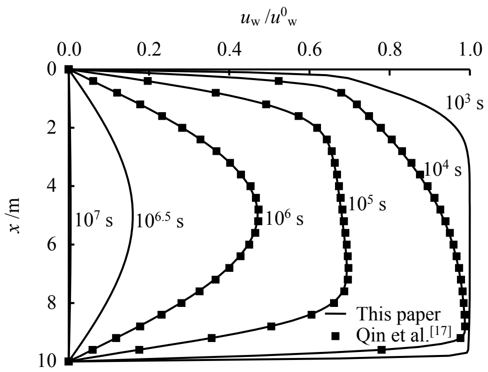

Fig. 2 Distribution of pore water pressure along the depth at different times

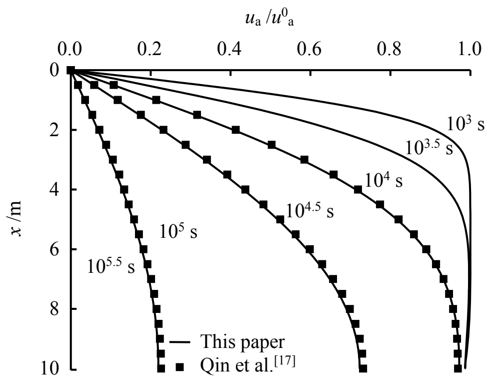

Fig. 3 Distribution of pore air pressure along the depth at different times

the two results are in good agreement with each other. And the slight difference in some points may result from the selection of different parameters when using the Crump's method to perform the inverse Laplace transform in Qin et al. [17]. Therefore, according to the comparison, the homogeneous boundary conditions calculated by the analytical method in this paper can be verified.

6.2 Effects of exponential boundary conditions

Mei et al.[29] firstly put forward the exponential boundary condition to analyze the consolidation of saturated soil, which solved the contradiction between homogeneous boundary condition and initial condition at time of "0". Afterwards, the researchers[26–28] introduced it into the study of consolidation of unsaturated soil. By improving the rationality of exponential boundary conditions at the impermeable boundary in references[26–28], this paper assumes a representative condition, that the pore water pressure and pore gas pressure at the top surface of the unsaturated soil, the pore water pressure and the gradient of pore gas pressure at the bottom surface are all exponential functions. The effect of exponential non-homogeneous boundary conditions on the consolidation process of unsaturated soil is analyzed. The boundary conditions are specifically expressed as

$$\left.\begin{array}{l} u_{\mathrm{w}}(0,t)=u_{\mathrm{w}}^{0}\mathrm{e}^{-\lambda_1 t},\ \ u_{\mathrm{a}}(0,t)=u_{\mathrm{a}}^{0}\mathrm{e}^{-\lambda_2 t} \\ u_{\mathrm{w}}(H,t)=u_{\mathrm{w}}^{0}\mathrm{e}^{-\lambda_3 t},\ \ u_{\mathrm{a},x}(H,t)=u_{\mathrm{a}}^{0}\mathrm{e}^{-\lambda_4 t} \end{array}\right\} \tag{62}$$

where λ_1, λ_2, λ_3 and λ_4 are the normal numbers (s^{-1}) to be given, and their values determine the growth rate of the boundary exponential function. When λ_1, λ_2 and λ_3 approach infinity, the pore pressure on the boundary quickly decreases to 0, becoming a permeable boundary. When λ_4 tends to infinity, the gradient of pore air pressure on the boundary rapidly decreases to zero, developing into an impermeable boundary, which is essentially different from the expressions of the impermeable boundary conditions proposed in references [26–28].

Figures 4 and 5 show the distribution of pore water pressure and pore gas pressure with depth at different times in the case of exponential boundary condition (62) and homogeneous boundary condition (61), respectively. Comparison analysis demonstrates that, when $\lambda_1=\lambda_2=\lambda_3=\lambda_4=10^2\ \mathrm{s}^{-1}$, the result of the exponential boundary condition equation (62) is almost the same as that of the homogeneous boundary condition equation (61). In this case, the exponential boundary condition is equivalent to the homogeneous boundary conditions in Eq. (61), and the consistency between the results from the two types of boundary conditions can used to verify analytical solutions under exponential boundary conditions. In this case, the consolidation process can be seen from the figure, due to the air-tight bottom surface at the initial moment, the pore water pressure at the top surface dissipates faster than that at the bottom surface. After the pore gas pressure dissipates (t>$10^{5.5}$s), the pore water pressure distributes symmetrically at the depth of 5 m and gradually dissipates to 0 on both sides. The pore gas continues to dissipate towards the surface, and the deeper the soil layer is, the longer time it takes to dissipate. At the beginning, the pore gas pressure at the bottom of the soil layer descends slightly, which can be explained that after the pore water on the bottom surface dissipates, the volume of pore air increases, leading to a reduction in the pressure.

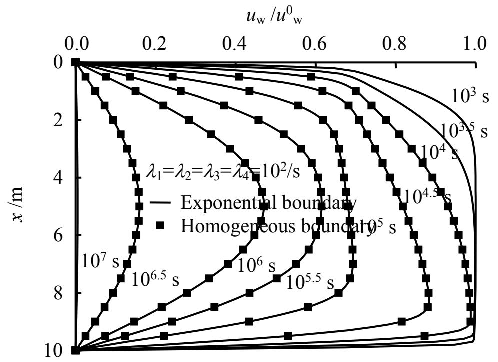

Fig. 4 Distribution of pore water pressure along the depth at different times with different boundary conditions

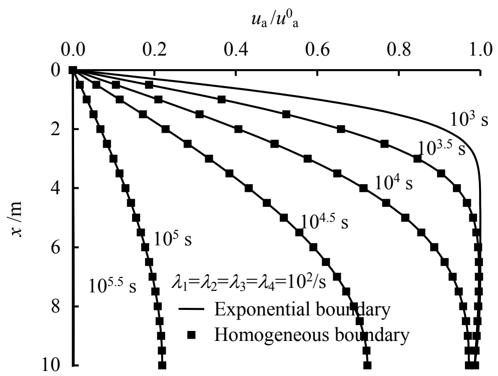

Fig. 5 Distribution of pore air pressure along the depth at different times with different boundary conditions

Figs. 6 and 7 represent the variations of pore water pressure and pore gas pressure over time at a depth of 5 m with different λ_1, respectively. $\lambda_2=\lambda_3=\lambda_4=10^2\ \mathrm{s}^{-1}$ means that the surface of the unsaturated soil layer is permeable, and the bottom surface is permeable for water but impermeable for air. The different values of λ_1 indicate that there are various impediment degrees of pore water dissipation on the surface of soil layer. As seen in Fig. 6, the pore water pressure at the initial time is not affected by the value of λ_1, but when the time is longer, the value of λ_1 becomes smaller, consequently, a greater impediment degree of pore water dissipation occurs, incurring a slower dissipation rate of pore water pressure. Although the value of λ_1 affects the pore water pressure, it has almost no effect on the pore gas pressure. Fig. 8 shows that the settlement of the soil varies with time. Affected by the dissipation process of pore water pressure, the settlement of the soil is not affected by λ_1 at the initial time, but the smaller the λ_1 at later stage, the longer it takes for the settlement to reach stability.

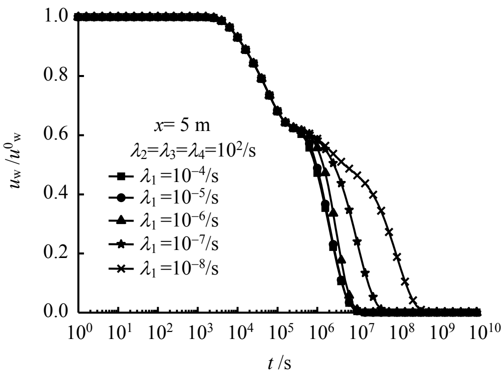

Fig. 6 Variation of pore water pressure versus time at x= 5 m under different values of λ_1

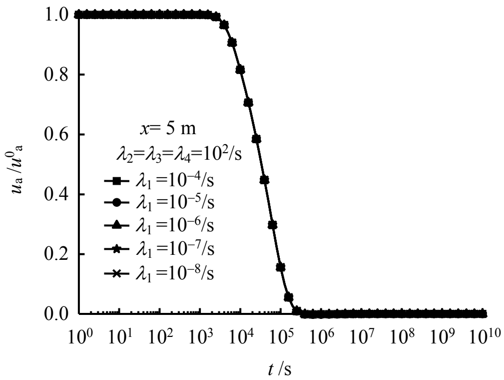

Fig. 7 Variation of pore air pressure versus time at x=5 m with different values of λ_1

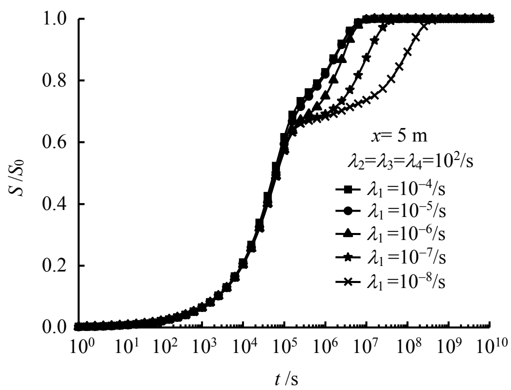

Fig. 8 Variation of surface settlement versus time with different values of λ_1

Figures. 9 and 10 display the variations of pore water pressure and pore gas pressure with time at a depth of 5 m with different λ_2, respectively. λ_1=λ_3=λ_4 10^2 s^{-1} means that both sides of the unsaturated soil layer are permeable for water and the bottom surface is impermeable for air. The different values of λ_2 represent the different degrees of retardation of pore gas dissipation on the surface of soil layer. As can be observed from Fig. 10, the smaller the λ_2, the slower the pore gas pressure dissipates. Meanwhile a smaller value of λ_2 will result in a slower dissipation in pore water pressures at the initial time, which is due to the coupling of the dissipation process of pore water pressure and pore gas pressure. Fig. 11 shows that the settlement of the soil varies with time. The smaller the λ_2 is, the slower the growth rate of the soil settlement is at the initial stage, but the time to reach the settlement stability is almost the same.

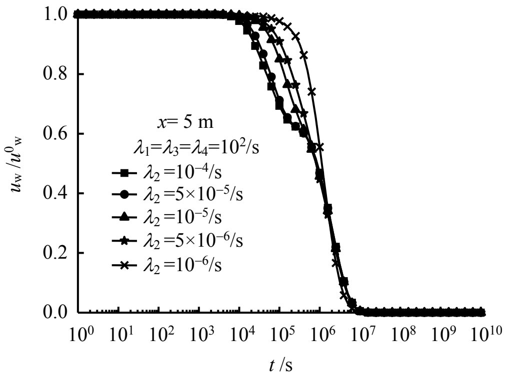

Fig. 9 Variation of pore water pressure versus time at x=5 m under different values of λ_2

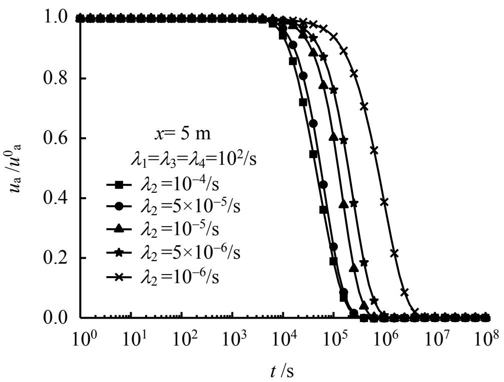

Fig. 10 Variations of pore air pressure versus time at x=5 m under different values of λ_2

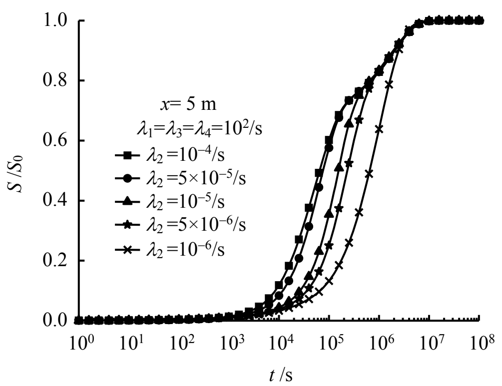

Fig. 11 Variations of surface settlement versus time under different values of λ_2

Figures 12 and 13 illustrates the variations of pore water pressure and pore gas pressure with time at the depth of 5 m with different λ_4, respectively. λ_1=λ_2=λ_3= 10^2 s^{-1} means that both sides of the unsaturated soil layer are water-permeable and the surface is air-permeable for air. According to the Fick law of gas, the exponential boundary condition of pore air pressure gradient on the bottom surface of Eq. (62) actually represents the equivalent flow rate of external gas entering the soil from the bottom of unsaturated soil layer. The various values of λ_4 represent the amount of gas entering the soil per unit time, and the entry of gas into the soil will cause the increase of pore gas pressure and pore water pressure. As can be seen from Figs. 12 and 13, the larger the λ_4, the less the gas enters the soil at the boundary at the beginning, the smaller the increase of pore gas pressure and pore water pressure, and when the time is longer, the flow rate of bottom gas into the soil decreases. Under the influence of the permeable boundary, the pore water pressure and pore gas pressure gradually dissipate to 0. From the previous analysis, it can be seen that at $\lambda_4=10^2\ \mathrm{s}^{-1}$, the bottom surface of the soil is close to the impermeable boundary, and the pore water pressure and pore gas pressure will not increase. Fig. 14 also shows that the soil settlement changes with time. Due to the increase of the initial pore pressure, the soil expands at first, and with the dissipation of pore water pressure and pore air pressure, the soil settlement gradually decreases to a stable value.

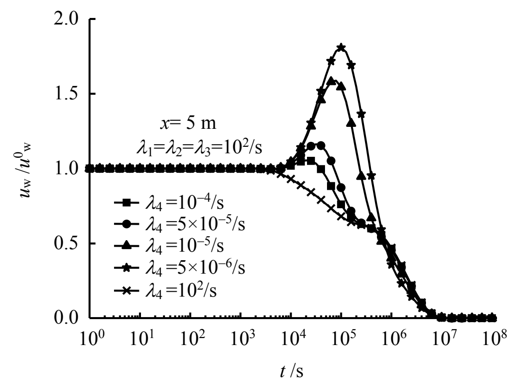

Fig. 12 Variation of pore water pressure versus time at x=5 m under different values of λ_4

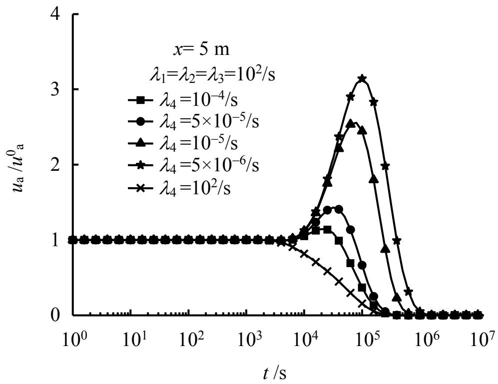

Fig. 13 Variation of pore air pressure versus time at x=5 m under different values of λ_4

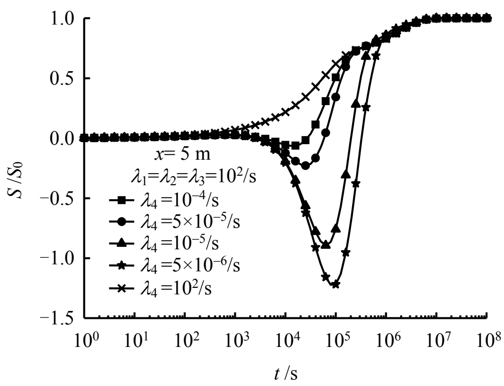

Fig. 14 Variation of surface settlement versus time under different values of λ_4

7 Conclusion

Based on the one-dimensional consolidation equation of unsaturated soil proposed by Fredlund et al, the analytical solution of one-dimensional consolidation of single-layer unsaturated soil under non-homogeneous mixed boundary conditions is given by using homogenization of non-homogeneous boundary conditions and eigenfunction expansion method. The proposed method is verified according to the solution of the homogeneous boundary, and the non-homogeneous boundary conditions which vary exponentially with time, are given. By changing the values of λ_1, λ_2 and λ_4, the variations of pore water pressure, pore air pressure and soil settlement are analyzed. The calculation results show that the larger the λ is, the closer the exponential boundary is to the homogeneous boundary condition, and the value of λ determines the change rates of pore water pressure, pore gas pressure or its gradient at the boundary, which has a great influence on the consolidation process of unsaturated soil.

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