Rock and Soil Mechanics

Volume 41 | Issue 12 Article 1

4-29-2021

Variable-order fractional damage creep model based on equivalent viscoelasticity for rock

De-jian LI
School of Mechanics and Civil Engineering, China University of Mining and Technology (Beijing), Beijing 100083, China

Xiao-lin LIU
School of Mechanics and Civil Engineering, China University of Mining and Technology (Beijing), Beijing 100083, China

Chao HAN
School of Mechanics and Civil Engineering, China University of Mining and Technology (Beijing), Beijing 100083, China

Custom Citation
LI De-jian, LIU Xiao-lin, HAN Chao, . Variable-order fractional damage creep model based on equivalent viscoelasticity for rock[J]. Rock and Soil Mechanics, 2020, 41(12): 3831-3839.

Rock and Soil Mechanics 2020 41(12): 3831−3839 ISSN 1000−7598
https: //doi.org/10.16285/j.rsm.2020.5419 rocksoilmech.researchcommons.org/journal

Variable-order fractional damage creep model based on equivalent viscoelasticity for rock

LI De-jian[1, 2], LIU Xiao-lin[1, 2], HAN Chao[1, 2]

1. State Key Laboratory for Geomechanics and Deep Underground Engineering, China University of Mining and Technology (Beijing), Beijing 100083, China
2. School of Mechanics and Civil Engineering, China University of Mining and Technology (Beijing), Beijing 100083, China

Abstract: Based on the equivalent viscoelasticity between the fractional Zener model and the time-varying viscosity Zener model, the significance of the relaxation time in the evolution of rock rheological viscoelasticity is highlighted. As the relaxation time approaches infinity, the evolution of viscosity during creep is equivalent to that during relaxation. Accordingly, the damage factor related to relaxation time is established to explain the physical significance of damage factor in rock rheological deformation. In specific, the function of variable order is constructed by using relaxation time and the variable-order fractional damage creep model is introduced. Furthermore, the variable-order fractional damage creep model is extended to the triaxial status. Based on the triaxial sandstone creep experimental data, the applicability and rationality of the variable-order fractional damage creep model in the triaxial state are verified. The results show that the developed variable-order fractional damage creep model and the creep experimental data are in good agreement, indicating that the damage creep model can be used to describe the nonlinear mechanical behavior of complete creep process. The effectiveness of the model fitting parameters is analyzed and verified, which encourages the applicability of the introduced model in other complex stress issues.

Keywords: rock rheology; equivalent viscoelasticity; relaxation time; evolution of damage; variable-order fractional damage creep model

1 Introduction

Rock rheology is a common phenomenon in underground engineering, which involves time-dependent deformation process of material[1−3]. In order to accurately predict and judge the creep failure time of rock, numerous element models have been applied in the description of creep mechanical behavior[4−6], e.g., classical Maxwell model, Zener model, Burgers model and Nishihara model etc. However, the element model is an integer order constitutive model in essence, which cannot better reflect the history-dependence of deformation. Due to the unique time memory of fractional-order calculus theory, fractional creep model has been attached great attentions by many scholars and a series of achievements have been achieved in recent years[7−14]. In 1978, Koeller et al.[7] replaced the traditional Newton dashpot in element model with the fractional dashpot constructed by R-L fractional operator. Yin et al.[8−9] presented a fractional Bingham model to describe the creep behavior of soft clay, which was based on fractional calculus. Zhou et al.[10] proposed a novel creep constitutive model of salt rock to describe three creep stages by replacing traditional dashpot with fractional dashpot with varying parameters. Chen et al.[11] introduced a fractional element with varying parameters to describe the decaying behavior of properties of rock material after damage. The three creep deformation stages can be well simulated by the introduced fractional creep model. Wu et al.[12] proposed a fractional damage creep model to depict three creep deformation stages of salt rock. Based on Caputo fractional theory, Liu et al.[13] presented a time-varying viscosity fractional creep model to reflect the creep mechanical behavior of sandstone. Based on the fractal theory, Su et al.[14] constructed a fractional non-Newtonian model to describe the viscoelastic behavior of material.

However, during the creep process of rock, the material properties vary with time, indicating that the fraction order should theoretically be a time-varying variable. Using the specific property of Gamma function and discrete approximation of infinite series, Zhang et al.[15] put forward a variable-order fractional rock creep model and applied the proposed model into the prediction of deformation of tunnel rock. Wu et al.[16−17] presented a new variable-order fractional rock creep model. His rock creep model took the deformation memory and time

Received: 13 April 2020 Revised: 29 May 2020
This work was supported by the National Key R&D Program of China(2016YFC0600901), the Fundamental Research Funds for the Central Universities (2010YL14) and the National Natural Science Foundation of China(41572334, 11572344).
First author: LI De-jian, male, born in 1966, PhD, Professor, mainly engaged in rock mechanics and numerical simulation research. E-mail: ldjcumtb@163.com
Corresponding author: LI Xiao-lin, male, born in 1994, PhD candidate, mainly engaged in rock mechanics rheological experiments and theoretical calculations. E-mail: 1403934802@qq.com

memory of each stage of rock creep into consideration. Generally, most of fractional creep models were based on constant-order fractional calculus and thus the physical meaning of fractional order cannot be clearly interpreted. Applying the variable-order fractional theory into rock creep model has gradually attracted many scholars' attentions[18–22]. When the variable-order fractional theory is applied in the construction of creep model, based on the different experimental conditions and time effect, Su et al.[18] presented a tri-axial variable-order fractional creep model from the view of analytic solutions. His new model considered the deformation behavior of three creep stages by introducing the variable-order fractional element. Based on the variable-order fractional theory and time effect, Wu et al.[19] proposed a modified variable-order Maxwell rock creep model, which has been verified by experimental creep data. However, when the fractional varying-order function is unknown, scholars use the finite difference method to discretize the variable order fractional equation so that the numerical calculations can be carried out. Bouras et al.[20] constructed a variable-order fractional creep model of concrete by conducting the numerical calculation based on finite difference method. The relationship between the order of fractional derivative and temperature were calibrated by experimental data. Sun et al.[21] used the finite difference method to discretize the variable-order fractional constitutive equation, and then established the relationship between the order of fractional derivative and variable according to the test conditions, so as to carry out the numerical calculation of the creep model. Based on variable-order fractional theory, Ramirez et al.[22] also presented a viscoelastic model. However, reviewing the application research of variable-order fractional calculus, the variable-order fractional rock creep model with clear physical meaning has been rarely reported and the study of variable-order fractional creep model under tri-axial status is also relatively insufficient.

In order to enrich the theory and engineering application of fractional creep model of rock, in this study, based on the equivalence of Zener rheological model, a variable-order fractional creep model related to relaxation time is established and the variable-order fractional creep model under tri-axial statue is proposed. Finally, based on the experimental creep data of sandstone under uniaxial and tri-axial conditions, the applicability and rationality of the model are verified.

2 Equivalent viscoelasticity of Zener model

Rocks show obvious viscoelastic properties during the rheological process. In order to better describe the viscoelastic mechanical property of rock materials in the rheological process, the classical Zener element model is selected as an important component of construction of rheological model[1]. In describing the material viscoelastic properties during rheological process, the dashpot element in Zener model can be replaced by fractional dashpot[7] and time-varying dashpot[1], and therefore the fractional Zener model (FZ) and time-varying viscosity Zener model (TVZ) are obtained. Both creep equations can be derived respectively from FZ model and TVZ model under constant stress. Since both creep equations can describe the viscoelastic behavior in the creep process, the equivalence between creep equations of FZ and TVZ model is employed to describe the viscous flow in the rheological process of materials. Therefore, the significance of relaxation time in the rheological process is highlyghted. The relaxation time can be further applied to the construction of rheological model.

2.1 FZ model and TVZ model

As shown in Fig.1, fractional Zener model is composed of fractional Maxwell model and one spring element in parallel and the constitutive equation of classical Koeller dashpot is expressed as follows[7]:

$$\sigma(t) = E\tau^{\alpha} \frac{\mathrm{d}^{\alpha} \varepsilon(t)}{\mathrm{d}t^{\alpha}} \tag{1}$$

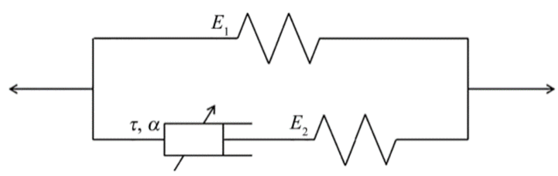


Fig. 1 Fractional Zener element model

where $\sigma(t)$ is stress and $\varepsilon(t)$ is strain. When $\alpha = 0$, Eq. (1) describes pure solid body. When $\alpha = 1$, Eq. (1) represents pure viscosity fluid body. When $0< \alpha <1$, Eq. (1) describes viscoelastic behavior of material. E_1 and E_2 represent elastic modulus of Zener model and fractional Maxwell model, respectively. The τ represents relaxation time and characterizes the internal time scale of relaxation phenomenon of material ($\tau = \eta/E$). And the α represents the order of fractional derivative. When the relaxation time τ is introduced to describe the viscoelasticity of dashpot, in order to simplify the calculation and reduce the parameters, this paper refers to reference[7] and assumes E is equal to E_2.

Based on the stress−strain relationship of elements in series and parallel, the constitutive model of FZ model can be obtained:

$$\sigma(t)+\tau^{\alpha}D^{\alpha}\sigma(t)=(E_1+E_2)\tau^{\alpha}D^{\alpha}\varepsilon(t)+E_1\varepsilon(t) \tag{2}$$

where E_1 is elastic modulus.

The Riemann-Liouville fractional operator[23–24] used in this paper is expressed as follows:

$$D^{\alpha}f(t)=\begin{cases}\dfrac{\mathrm{d}^m}{\mathrm{d}t^m}\displaystyle\int_0^t\frac{\xi^{m-\alpha-1}}{\Gamma(m-\alpha)}f(\xi)\mathrm{d}\xi, & m-1<\alpha<m\\[2ex] \dfrac{\mathrm{d}^m f(t)}{\mathrm{d}t^m}, & \alpha=m\end{cases} \tag{3}$$

The expression of R-L fractional operator under Laplace transform is obtained:

$$L(D^{\alpha}f(t))=s^{a}\tilde{f}(s) \tag{4}$$

where L represents Laplace forward transform and s is expression of time in frequency domain.

Then the Laplace forward transform is applied in FZ constitutive model, when $\sigma=\sigma_0 H(t)$, the creep compliance of FZ model can be derived as follows:

$$J_{\mathrm{FZ}}(s)=L^{-1}\left[\frac{\tau^{\alpha}s^{\alpha-1}+s^{-1}}{(E_1+E_2)\tau^{\alpha}s^{\alpha}+E_1}\right]=\frac{1}{E_1+E_2}+\frac{E_2}{E_1+E_2}L^{-1}\left[\frac{s^{-1}}{(E_1+E_2)\tau^{\alpha}s^{\alpha}+E_1}\right] \tag{5}$$

where $H(t)$ is Heaviside function.

According to the Laplace inverse transform and convergence of Mittag-Leffler function[26], Eq. (5) can be simplified into

$$J_{\mathrm{FZ}}(t)=\frac{1}{E_1}-\frac{E_2}{E_1(E_1+E_2)}M_{\alpha,1}\left[-\frac{E_1}{E_1+E_2}\left(\frac{t}{\tau}\right)^{\alpha}\right] \tag{6}$$

Similarly, the Laplace forward transform and Mellin transform are applied on FZ constitutive model[23, 26–27], when $\varepsilon(t)=\varepsilon_0 H(t)$, the relaxation modulus of FZ model can be written as follows:

$$G_{\mathrm{FZ}}(t)=E_1+\frac{E_2}{\alpha}H_{12}^{11}\left[\frac{t}{\tau}\middle|\begin{matrix}0 & 1/\alpha\\ 0 & 1/\alpha;\ (0,1)\end{matrix}\right] \tag{7}$$

where H_{12}^{11} is Fox-H function [23, 26–27].

And based on the convergence of Mittag-Leffler function, Eq. (7) can be simplified into

$$G_{\mathrm{FZ}}(t)=E_1+E_2M_{\alpha,1}\left[-\left(\frac{t}{\tau}\right)^{\alpha}\right] \tag{8}$$

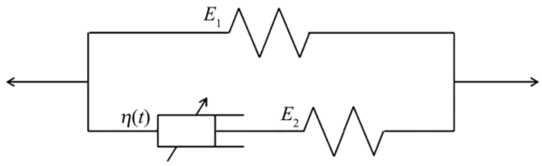


Fig. 2 Time-varying viscosity Zener element model

As shown in Fig.2, the time-varying viscosity Zener model (TVZ) is composed of the time-varying Maxwell model and the spring element in parallel. It is well-known that during the process of rheology, the viscosity of material changes with time. For better exhibiting the viscosity flow in the process of rheology, the traditional Newton dashpot is replaced by the time-varying viscosity dashpot. Based on the stress−strain relationship of elements in series and parallel, the constitutive model of TVZ model is yielded as follows:

$$\sigma(t)+\frac{\eta(t)}{E_2}\dot{\sigma}(t)=E_1\varepsilon(t)+\eta(t)\frac{E_1+E_2}{E_2}\dot{\varepsilon}(t) \tag{9}$$

where $\eta(t)$ represents time-varying viscosity; $\dot{\sigma}(t)$ and $\dot{\varepsilon}(t)$ are the first derivative of stress and strain, respectively.

Based on the initial boundary conditions, i.e., $\sigma_0=\varepsilon_0(E_1+E_2)$, when $\sigma(t)=\sigma_0$, the creep compliance of TVZ model can be derived by the method of variable separation.

$$J_{\mathrm{TVZ}}(t)=\frac{1}{E_1}-\frac{E_2}{E_1(E_1+E_2)}\exp\left[-\int_0^t\frac{E_2E_1}{\eta(\tau)(E_1+E_2)}\mathrm{d}\tau\right] \tag{10}$$

In a similar way, when $\varepsilon(t)=\varepsilon_0$, based on the method of variable separation and initial boundary conditions, the relaxation modulus of TVZ model can be obtained as follows:

$$G_{\mathrm{TVZ}}(t)=E_1+E_2\exp\left[-\int_0^t\frac{E_2}{\eta(\tau)}\mathrm{d}\tau\right] \tag{11}$$

2.2 Effect of relaxation time on equivalent viscoelasticity

As mentioned in Section. 2.1, both FZ model and TVZ model can depict the viscoelastic mechanical behavior of materials in process of rheology. By setting the creep and relaxation response of FZ model are equal to the creep and relaxation response of TVZ model, the time-varying viscosity function can be obtained and the physical meaning of fractional viscoelasticity can be well understood. In this section, this equivalent idea will be applied in the Zener rheological model. First, by making Eq. (6) equal to Eq. (10), i.e., $J_{\mathrm{FZ}}(t)=J_{\mathrm{TVZ}}(t)$, the time-varying viscosity equation in creep process can be obtained as follows:

$$\eta_c(t) = E_2 \tau^{\alpha} t^{1-\alpha} \frac{M_{\alpha,1}\left(-E_0\left(\frac{t}{\tau}\right)^{\alpha}\right)}{M_{\alpha,\alpha}\left(-E_0\left(\frac{t}{\tau}\right)^{\alpha}\right)} \tag{12}$$

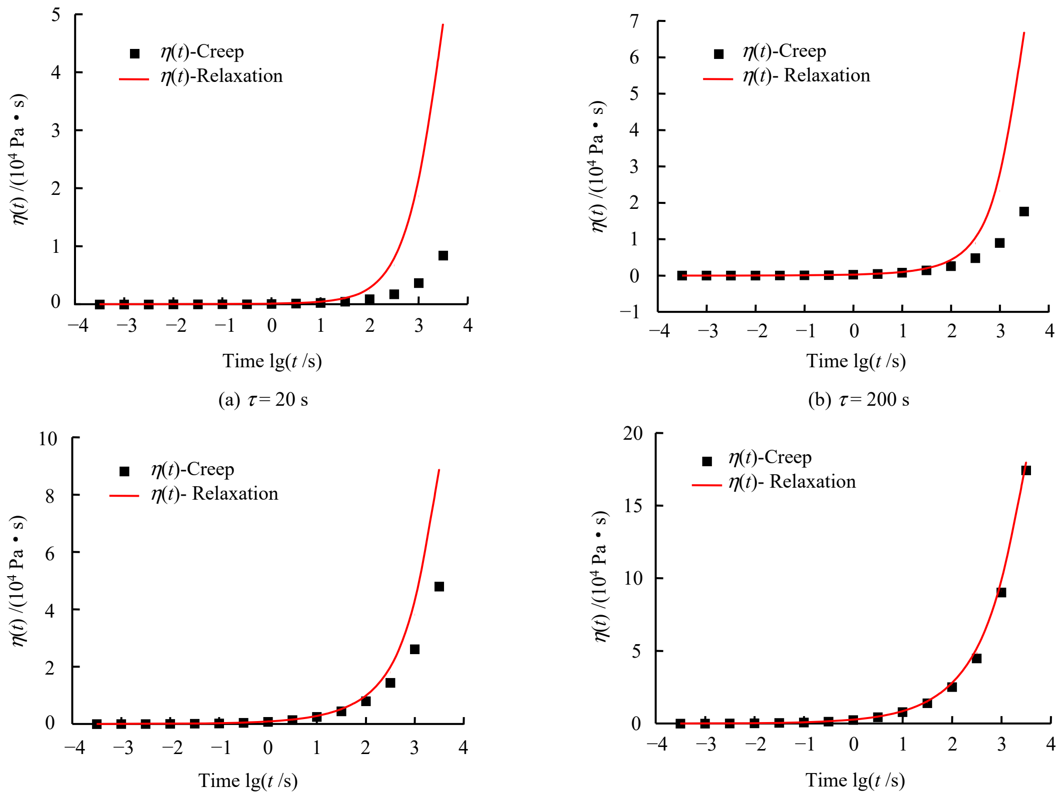
where $M_{\alpha,1}$ is Mittag-Leffler function with two parameters and $\eta_c(t)$ represents viscosity in creep process. $E_0 = \dfrac{E_1}{E_1 + E_2}$.

Similarly, the equivalent viscoelasticity of rheology is applied in Zener relaxation model and by setting Eq. (7) equal to Eq. (11), i.e., $G_{FZ}(t) = G_{TVZ}(t)$, the time-varying viscosity equation in the process of relaxation can be deduced as follows:

$$\eta_r(t) = E_2 \tau^{\alpha} t^{1-\alpha} \frac{M_{\alpha,1}\left(-\left(\frac{t}{\tau}\right)^{\alpha}\right)}{M_{\alpha,\alpha}\left(-\left(\frac{t}{\tau}\right)^{\alpha}\right)} \tag{13}$$

where $\eta_r(t)$ represents the viscosity in the process of relaxation.

As illustrated in Fig.3, when relaxation time increases gradually, the viscosity coefficient of FZ model gradually approaches the viscosity coefficient of TVZ model. It is shown that the relaxation time can control the viscosity evolution of Zener model during the rheological process, and the idea of rheological equivalent viscoelasticity is also verified. As a consequence, the effect of relaxation time should be taken into account in the later rheological model construction. It is also shown in Fig. 3 that the viscosities of FZ model and TVZ model increase with the increase of time. Meanwhile, viscosity plays an important role in the resistance to rheological deformation of materials. As the material viscosity is gradually consumed, the proportion of viscosity in the properties of materials decreases gradually.



(a) τ = 20 s (b) τ = 200 s (c) τ = 2 000 s (d) τ = 20 000 s

Fig. 3 Relationship between viscosity coefficient of FZ model and that of TVZ model under different relaxation time where the fractional order is 0.5

3 Damage creep model based on variable-order fractional theory

3.1 Damage factor considering relaxation time

It can be concluded from Section 2 that relaxation time is a significant factor affecting the evolution of viscoelasticity in the rheological process of materials, which deserves great concerns in the total process of rheology. During the process of rheological deformation, from the initial load to the final damage, the characterization of material damage is necessary to establish creep model. Based on the significance of relaxation time in the rheological process, a damage factor with physical significance based on relaxation time is proposed in this section.

$$D = 1 - \exp\left(-\left(\frac{t}{\tau}\right)\right) \tag{14}$$

where D is damage factor describing failure characteristics of materials.

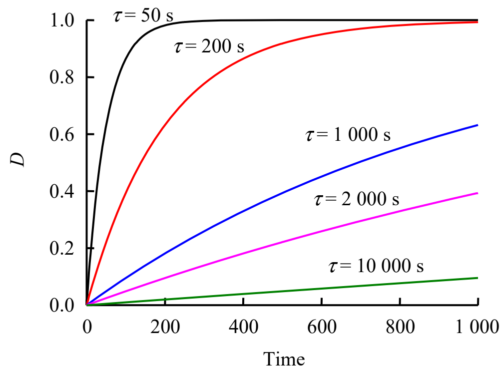


Fig. 4 Introduced damage factor based on relaxation time

As shown in Fig. 4, the longer the relaxation time is, the more time is needed for damage development and the gentler the damage degree is. It suggests that it is reasonable to consider the development of relaxation time in damage factor. With the increase of time, the damage factor of materials keeps increasing, which is also consistent with the crack propagation when the failure occurs inside the rock materials. The cracks keep increasing with the increase of time. This founding physically verifies the rationality of the proposed damage factor.

3.2 Variable-order fractional damage creep model considering relaxation time

Rock creep is a time-dependent deformation process, which commonly involves decaying, steady and accelerating creep three stages[1]. The nonlinear behavior is more obvious in accelerating creep stage and the rock damage develops rapidly in this stage as well. Therefore, in order to better describe the nonlinear mechanical behavior of three creep deformation stages of rock, the important role of relaxation time is considered in the viscoelastic evolution and damage development process of materials. In this section, a nonlinear damage creep model is proposed by combining variable-order fractional Zener model with damage factor that considers relaxation time in series, which is demonstrated in Fig. 5.

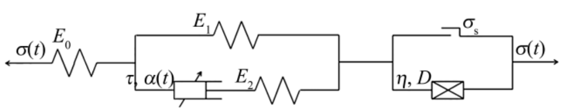


Fig. 5 Variable-order fractional damage creep model

First, based on the R-L fractional theory, when stress is a constant value σ_0, constant-order fractional Zener creep model is expressed as follows:

$$\varepsilon_{\mathrm{FZ}}(t) = \frac{\sigma_0}{E_1} - \frac{\sigma_0 E_2}{E_1(E_1+E_2)} M_{\alpha,1}\left[-\frac{E_1}{E_1+E_2}\left(\frac{t}{\tau}\right)^{\alpha}\right] \tag{15}$$

When rock creep deformation occurs, the viscosity of material is constantly consumed over time to resist deformation, which leads to the continuous change of the viscoelasticity proportion within material. Hence, in order to clearly express the evolution of mechanical property in creep process through fractional theory, the order of fractional derivative should be assumed as a time-dependent function. As aforementioned, considering the importance of relaxation time in the process of viscoelasticity evolution and damage development of material, a varying-order function within fractional theory is presented as follows:

$$\alpha(t) = \exp\left(-\left(\frac{t}{\tau}\right)\right) \tag{16}$$

Substituting the varying-order function into Eq. (15) leads to the variable-order fractional Zener creep model:

$$\varepsilon_{\mathrm{VFZ}}(t) = \frac{\sigma_0}{E_0} + \frac{\sigma_0}{E_1} - \frac{\sigma_0 E_2}{E_1(E_1+E_2)} M_{\alpha(t),1}\left[-\frac{E_1}{E_1+E_2}\left(\frac{t}{\tau}\right)^{\alpha(t)}\right] \tag{17}$$

where E_0 is the elastic modulus.

When creep deformation develops toward last stage, damage factor will play a dominant role. The proposed novel damage body that considering relaxation time will be applied in the description of accelerating creep stage of rock, whose expression is shown as follows:

$$\varepsilon_a(t) = \frac{\sigma_0 - \sigma_s}{\eta(1-D)} \tag{18}$$

where $\varepsilon_a(t)$ is the strain resulted from damage body and plastic switch.

Hence, the variable-order fractional damage creep model can be finally expressed as follows:

$$\varepsilon(t) = \begin{cases} \dfrac{\sigma_0}{E_0} + \dfrac{\sigma_0}{E_1} - \dfrac{\sigma_0 E_2}{E_1(E_1+E_2)} M_{\alpha(t),1}\left[-\dfrac{E_1}{E_1+E_2}\left(\dfrac{t}{\tau}\right)^{\alpha(t)}\right], \\ \qquad \sigma_0 \leqslant \sigma_s \\ \dfrac{\sigma_0}{E_0} + \dfrac{\sigma_0}{E_1} - \dfrac{\sigma_0 E_2}{E_1(E_1+E_2)} M_{\alpha(t),1}\left[-\dfrac{E_1}{E_1+E_2}\left(\dfrac{t}{\tau}\right)^{\alpha(t)}\right] + \\ \dfrac{\sigma_0 - \sigma_s}{\eta(1-D)}, \quad \sigma_0 > \sigma_s \end{cases} \tag{19}$$

3.3 Variable-order fractional damage creep model under tri-axial status

Since the rock mass is in tri-axial compressive state under engineering conditions, for better describing the mechanical behavior of rock creep under tri-axial state, in this study, the variable-order fractional damage creep model is established under tri-axial state, which can accurately predict and simulate the mechanical behavior of rock creep under tri-axial state in following section.

First, the total strain of rock is assumed as $\varepsilon_{ij}(t)$, based on strain superposition principle, we can obtain:

$$\varepsilon_{ij}(t) = \varepsilon_{ij}^{\mathrm{S}}(t) + \varepsilon_{ij}^{\mathrm{VFZ}}(t) + \varepsilon_{ij}^{\mathrm{ND}}(t) \tag{20}$$

where $\varepsilon_{ij}^{\mathrm{S}}(t)$, $\varepsilon_{ij}^{\mathrm{VFZ}}(t)$ and $\varepsilon_{ij}^{\mathrm{ND}}(t)$ are strains of spring, VFZ model and nonlinear damage body, respectively.

The relationships among stress tensor, strain tensor, spherical tensor of stress and spherical tensor of strain are shown as follows:

$$\left.\begin{aligned} \boldsymbol{\sigma}_{ij} &= \boldsymbol{S}_{ij} + \boldsymbol{\sigma}_{\mathrm{m}}\delta_{ij} \\ \boldsymbol{\varepsilon}_{ij} &= \boldsymbol{e}_{ij} + \boldsymbol{\varepsilon}_{\mathrm{m}}\delta_{ij} \end{aligned}\right\} \tag{21}$$

where σ_m and ε_m are the spherical tensor of stress and spherical tensor of strain, respectively; S_{ij} and e_{ij} are deviator tensor of stress and deviator tensor of strain, respectively; and δ_{ij} is Kronecker function[27].

The instantaneous elastic deformation behavior of initial creep stage can be depicted by spring element, whose expression of creep is

$$\varepsilon_{ij}^{\mathrm{S}}(t) = \frac{S_{ij}}{2G_0} \tag{22}$$

For the fractional Zener model, in the early creep stage, most of deformations are attributed to shear deformation and most of deformation characteristics are viscoelasticity evolution. Thus, tri-axial creep equation of fractional Zener model can be simplified into tri-axial creep equation of fractional Maxwell model, whose expression is written as

$$\boldsymbol{\varepsilon}_{ij}^{\mathrm{VFZ}}(t) = \frac{\boldsymbol{S}_{ij}}{2G_1} + \frac{\boldsymbol{\sigma}_{\mathrm{m}}\delta_{ij}}{3K} + \frac{\boldsymbol{S}_{ij}t^{\alpha(t)}}{2G\Gamma(1+\alpha(t))} \tag{23}$$

where G is the shear modulus and K is the bulk modulus.

For nonlinear damage body, based on the generalized plastic mechanics and associated flow rule[27], the creep equation of nonlinear damage body is expressed as follows:

$$\varepsilon_{ij}^{\mathrm{ND}}(t) = \frac{\exp(t/\tau)}{\eta}\left(g\left(\frac{F}{F_0}\right)\right)\frac{\partial f}{\partial \sigma_{ij}}t \tag{24}$$

where F is the yield function and F_0 is the initial value of yield function, in general, $F_0 = 1$. f is the plastic potential function. According to associated flow rule, $F = f$.

During the process of tri-axial creep of rock, the deviator tensor of stress generally plays a dominant role and the effect of spherical tensor of stress on creep deformation is relative weak. So the Misses criterion[27] is used in this paper, as shown below:

$$F = \sqrt{J_2} - \frac{\sigma_{\mathrm{s}}}{\sqrt{3}} \tag{25}$$

where J_2 is the second invariant in deviator tensor of stress.

Because tri-axial creep experiment is generally conventional tri-axial creep experiment, several initial conditions are given as

$$\left.\begin{aligned} &\sigma_2 = \sigma_3 < \sigma_1 \\ &\sigma_m = \frac{\sigma_1 + 2\sigma_3}{3} \\ &S_{11} = \sigma_1 - \frac{\sigma_1+\sigma_2+\sigma_3}{3} = \frac{2(\sigma_1-\sigma_3)}{3} \\ &\sqrt{J_2} = \sqrt{\frac{(\sigma_1-\sigma_2)^2 + (\sigma_1-\sigma_3)^2 + (\sigma_2-\sigma_3)^2}{6}} \end{aligned}\right\} \tag{26}$$

By substituting Eq. (25) into Eqs. (23) and (24), we can obtain:

$$\varepsilon_{ij}^{\mathrm{ND}}(t) = \frac{\sigma_1 - \sigma_3 - \sigma_{\mathrm{s}}}{3}t\exp\left(\frac{t}{\tau}\right) \tag{27}$$

Thus, the variable-order fractional damage creep equation under tri-axial state can be derived by substituting Eqs. (22)−(27) into Eq. (20):

$$\varepsilon_{11}(t) = \begin{cases} \dfrac{(\sigma_1-\sigma_3)}{3G_0} + \dfrac{(\sigma_1-\sigma_3)}{3G_1} + \dfrac{\sigma_1+2\sigma_3}{9K}\delta_{ij} + \\ \quad \dfrac{(\sigma_1-\sigma_3)t^{\alpha(t)}}{3G_1\Gamma(1+\alpha(t))}, \quad (\sigma_1-\sigma_3) < \sigma_{\mathrm{s}} \\ \dfrac{(\sigma_1-\sigma_3)}{3G_0} + \dfrac{(\sigma_1-\sigma_3)}{3G_1} + \dfrac{\sigma_1+2\sigma_3}{9K}\delta_{ij} + \\ \dfrac{(\sigma_1-\sigma_3)t^{\alpha(t)}}{3G_1\Gamma(1+\alpha(t))} + \dfrac{(\sigma_1-\sigma_3)-\sigma_{\mathrm{s}}}{3\eta}t\exp\left(\dfrac{t}{\tau}\right), \\ \quad (\sigma_1-\sigma_3) \geqslant \sigma_{\mathrm{s}} \end{cases} \tag{28}$$

4 Verification of variable-order fractional damage creep model

4.1 Verification of reasonability of tri-axial model

For verifying the applicability of proposed variable-order fractional damage creep model in the description of nonlinear mechanical behavior of rock creep, in this study, the creep experimental data of sandstone under conventional tri-axial creep experiments is selected from previous studies[28]. Sandstone samples are taken from a deep foundation pit in Chongqing and the long-term strength of sandstone sample is 51.11 MPa. A series of compressive creep experiments was conducted on sandstone samples under a confining pressure of 5 MPa by RLW-2000 apparatus. The specific experimental scheme is shown in Fig. 6.

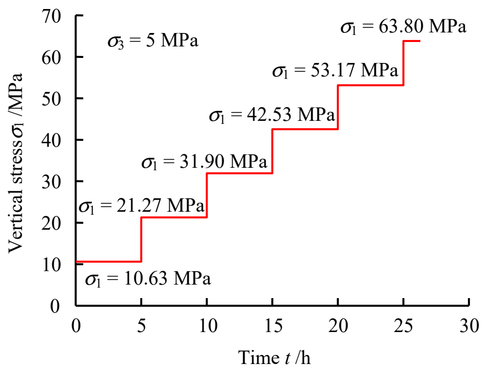
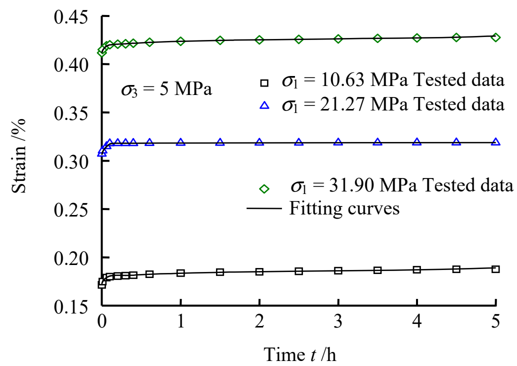


Fig. 6 Triaxial creep test scheme for sandstone

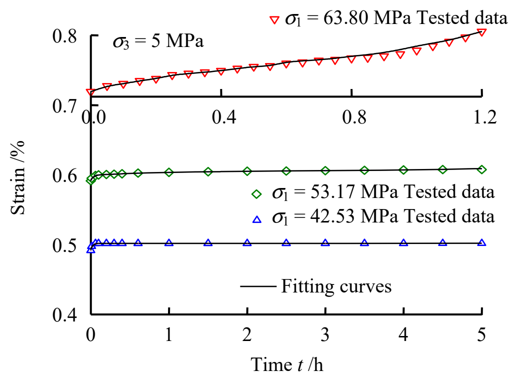
Based on the sandstone creep data under a confining pressure of 5 MPa, the tri-axial variable-order fractional damage creep model of Eq. (27) is adopted to analyse the experimental data. In order to concisely show the applicability of proposed model on tri-axial creep experimental data, tri-axial sandstone creep data under six different stress status (vertical pressures with 10.63, 21.27, 31.90, 42.53, 53.17 and 63.80 MPa and confining pressure with 5 MPa) are selected as fitting data. Figure 7 shows the high correspondence between experimental data and fitting curves. The model can not only describe the viscoelastic behaviors in decaying creep and steady creep stages, but also shows advantages in characterizing the nonlinear behavior in accelerating creep stage. As listed in Table 1, although the parameters under different stress levels vary at a certain degree, the variations of fitting parameters are insignificant. Especially, for a given rock, the parameters in the model can be assumed to be constant, which further verifies the applicability and reasonability of proposed tri-axial variable-order fractional damage creep model.

4.2 Validity analysis of model parameters

It can be known from above fitting results in Section 4.1, the proposed model is in good agreement with experimental data. The fitting parameters change regularly with the increase of stress level. It is worth noting that when



(a) Experimental data under confining pressure of 5 MPa and vertical pressures of 10.63, 21.27 and 31.90 MPa and curve-fit of proposed model



(b) Experimental data under confining pressures of 5 MPa and vertical pressures of 42.53, 53.17 and 63.80 MPa and curve-fit of proposed model

Fig. 7 Fitting curves of variable-order fractional damage creep model based on tri-axial creep experimental data

Table 1 Fitting parameters of creep model based on tri-axial creep data of sandstone

σ_1 /MPa	G_0 /GPa	G_1 /(GPa • h^{α})	K /GPa	τ	η /(GPa • h^{α})
10.63	0.043	0.189	0.071	0.420 6	—
21.27	0.079	5.643	0.152	0.340 1	—
31.90	0.120	0.890	0.253	0.403 2	—
42.53	0.150	7.725	0.349	0.113 7	—
53.17	0.186	1.774	0.450	0.450 4	—
63.80	0.214	0.642	0.546	0.346 7	2.027

stress levels are 21.27 and 42.53 MPa, comparing to other stress levels, abnormal variations appear in shear modulus G_1 and relaxation time τ, which is resulted from uncertainty and randomness of testing data and errors of fitting method. Therefore, the validity analysis of the obtained fitting parameters is reasonable, which contributes to the promotion and application of the variable-order fractional damage creep model.

As shown in Fig. 8, when the confining pressure is 5 MPa, shear modulus G_0 of the fitting parameter increases linearly with the increase of axial stress, and the change law conforms to the law that instantaneous elastic modulus

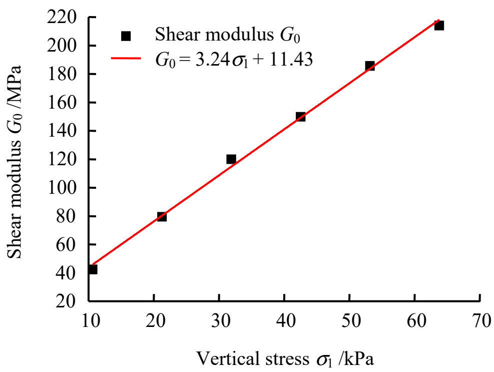


Fig. 8 Relationship between fitting parameters G_0 and vertical stress

increases with the increase of stress level[27]. The shear modulus under different stress levels can be obtained by substituting into the function $G_0 = 3.24\sigma_1 + 11.43$.

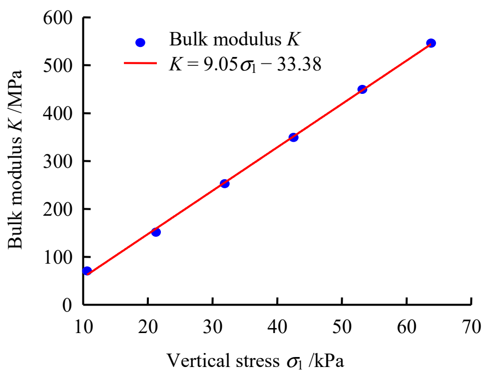


Fig. 9 Relationship between fitting parameters K and vertical stress

As shown in Fig. 9, when the confining pressure is 5 MPa, bulk modulus K of the model fitting parameter increases linearly with axial stress and bulk modulus under various stress levels can be calculated by function, $K = 9.05\sigma_1 - 33.38$.

5 Conclusions

By using the equivalence between fractional Zener model and time-varying viscosity Zener model, the importance of relaxation time in the evolution of rheological viscoelasticity is highlighted based on the idea of equivalent viscoelasticity.

Based on the relaxation time, a new damage factor dependent on relaxation time is proposed and its physical meaning in the process of rock damage is clearly explained, which can describe the nonlinear mechanical behavior of rock in the accelerating creep stage.

A varying-order function related to relaxation time is proposed and a uniaxial variable-order fractional damage creep model is constructed, and the uniaxial model is further extended to tri-axial variable-order fractional damage creep model.

Based on current experimental creep data of sandstone, the applicability and reasonability of the proposed model are verified. The new model can well fit three creep stages of sandstone and has a good correlation with the experimental data.

The validity of the model fitting parameters is analyzed and the mathematical expressions of shear modulus and bulk modulus with variation of stress levels are also provided. The proposed creep model can be better applied to other complex stress levels in further study.

References

[1] SUN Jun. Rheology of geomaterials and its engineering application[M]. Beijing: China Architecture and Building Press, 1990.

[2] ZHOU Wei-yuan. Advanced rock mechanics[M]. Beijing: China Water Conservancy and Hydropower Press, 2005.

[3] ZHU Sai-nan, YIN Yue-ping, LI Bin. Shear creep behavior of soft interlayer in Permian carbonaceous shale[J]. Rock and Soil Mechanics, 2019, 40(4): 1377−1386.

[4] YANG Xiu-rong, JIANG An-nan, JIANG Zong-bin, et al. Creep test and damage model of soft rock under water containing condition[J]. Rock and Soil Mechanics, 2018, 39(Suppl.1): 176−183.

[5] HE Li-jun, KONG Ling-wei, WU Wen-jun, et al. A description of creep model for soft soil with fractional derivative[J]. Rock and Soil Mechanics, 2011, 32(Suppl.2): 239−245.

[6] YU J, LIU G Y, CAI Y Y, et al. Time-dependent deformation mechanism for swelling soft-rock tunnels in coal mines and its mathematical deduction[J]. International Journal of Geomechanics, 2020, 20(3): 04019186.

[7] KOELLER R C. Applications of fractional calculus to the theory of viscoelasticity[J]. Journal of Applied Mechanics, 1984, 51(2): 299−307.

[8] YIN De-shun, REN Jun-juan, HE Cheng-liang, et al. A new rheological model element for geomaterials[J]. Chinese Journal of Rock Mechanics and Engineering, 2007, 26(9): 1899−1903.

[9] YIN D, ZHANG W, CHENG C, et al. Fractional time dependent Bingham model for muddy clay[J]. Journal of Non-Newtonian Fluid Mechanics, 2012, 187−188: 32−35.

[10] ZHOU H W, LIU D, LEI G, et al. The creep-damage model of salt rock based on fractional derivative[J]. Energies, 2018, 11(9): 2349−2358.

[11] CHEN L, WANG C P, LIU J F, et al. A damage-mechanism-based creep model considering temperature effect in granite[J]. Mechanics Research Communications, 2014, 56: 76−82.

[12] WU F, CHEN J, ZOU Q L. A nonlinear creep damage model for salt rock[J]. International Journal of Damage Mechanics, 2019, 28(5): 758−771.

[13] LIU Xiao-lin, LI De-jian, HAN C. A nonlinear damage creep model for sandstone based on fractional theory[J]. Arabian Journal of Geosciences, 2020, 13(6): 246−254.

[14] SU X, CHEN W, XU W. Characterizing the rheological behaviors of non-Newtonian fluid via a viscoelastic component: fractal dashpot[J]. Advances in Mechanical Engineering, 2017, 9(10): 1−12.

[15] ZHANG J Z, ZHOU X P, YIN P. Visco-plastic deformation analysis of rock tunnels based on fractional derivatives[J]. Tunnelling and Underground Space Technology, 2019, 85: 209−219.

[16] WU F, LIU J F, WANG Jun. An improved maxwell creep model for rock based on variable-order fractional derivatives[J]. Environmental Earth Sciences, 2015, 73(11): 6965−6971.

[17] WU F, CHEN J, ZOU Q L. A nonlinear creep damage model for salt rock[J]. International Journal of Damage Mechanics, 2019, 28(5): 758−771.

[18] SU Teng, ZHOU Hong-wei, ZHAO Jia-wei, et al. A creep model of rock based on variable order fractional derivative[J]. Chinese Journal of Rock Mechanics and Engineering, 2019, 38(7): 1355−1363.

[19] WU F, GAO R B, LIU J. et al. New fractional variable-order creep model with short memory[J]. Applied Mathematics and Computation, 2020, 380: 125278.

[20] BOURAS Y, DUŠAN ZORICA, TEODOR M, et al. A non-linear thermo-viscoelastic theological model based on fractional derivatives for high temperature creep in concrete[J]. Applied Mathematical Modelling, 2018, 55: 551−568.

[21] SUN H G, CHEN W, WEI H, et al. A comparative study of constant-order and variable-order fractional models in characterizing memory property of systems[J]. European Physical Journal Special Topics, 2011, 193(1): 185−192.

[22] RAMIREZ L, COIMBRA C. A variable order constitutive relation for viscoelasticity[J]. Annals of Physics, 2007, 16(8): 543−552.

[23] GUO Dun-ren. Method of mathematical physics[M]. Beijing: People's Education Press, 1978.

[24] MENG R, YIN D, LU S, et al. Variable-order fractional constitutive model for the time-dependent mechanical behavior of polymers across the glass transition[J]. The European Physical Journal Plus, 2019, 134: 376−385.

[25] XIAO R, SUN H, CHEN W. An equivalence between generalized Maxwell model and fractional Zener model[J]. Mechanics of Materials, 2016, 100: 148−153.

[26] BAGLEY R L, TORVIK P J. A theoretical basis for the application of fractional calculus to viscoelasticity[J]. Journal of Rheology, 1983, 27(3): 115−198.

[27] ZHENG Ying-ren, KONG Liang. Generalized plastic mechanics and its application[J]. Engineering Science, 2005, 7(11): 21−36.

[28] LIU Dong-yan, XIE Lin-jie, TUO Xiao-feng, et al. Creep properties of sandstone under different confining pressures and research on a nonlinear viscoelasto-plastic creep model[J]. Chinese Journal of Rock Mechanics and Engineering, 2017, 36(Suppl.2): 3705−3712.